



JEEEnius AI

PREVIOUS YEAR QUESTIONS · 2007–2025

# JEE Advanced

## Chapter-wise PYQs

Mathematics

**2007–2025** Questions from 19 years

Chapter-wise previous year questions with full solutions · Papers 1 & 2

**665**

QUESTIONS

**9**

FIGURES

**15**

CHAPTERS



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# JEE Advanced Chapter-wise PYQs · Mathematics

Chapter-wise previous year questions with full solutions · Papers 1 & 2

Digital edition · First published October 2026

## Published by

Grey Neural Technologies Private Limited

CIN U62011WR2026PTC296120

Registered office: Flat No. E-1/14, S.M. Nagar, Sarkarpool, South 24 Parganas, West Bengal 700143, India

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Typeset by JEEEnius · 665 questions · built 2026-10-16

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- **Chapter-wise, newest first.** Every chapter opens with its questions, latest year first, followed by an answer key and then full solutions. Attempt the questions before turning to the key.
- **Source line.** Each question shows where it was asked — year, paper and marking scheme, and its type when it is not a single-correct MCQ.
- **Weightage table.** Mathematics opens with a table of how often every chapter was asked, overall and in the last five years. Use it to plan revision.
- **More than an answer.** Solutions carry a difficulty rating, a suggested time and the topic; many add a quick approach, the common traps students fall into, and what to revise.
- **Ref numbers.** Each solution carries a Ref number. Quote it to report an error.

### Questions by year

|         |         |         |         |         |         |         |         |         |         |         |         |         |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| 2007 38 | 2008 34 | 2009 34 | 2010 36 | 2011 42 | 2012 35 | 2013 24 | 2014 46 | 2015 53 | 2016 32 | 2017 31 | 2018 35 | 2019 34 |
| 2020 34 | 2021 33 | 2022 29 | 2023 31 | 2024 33 | 2025 31 |         |         |         |         |         |         |         |



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# Contents

|                                         |          |                                         |    |     |
|-----------------------------------------|----------|-----------------------------------------|----|-----|
| <b>JEE Advanced 2007–2025</b>           | <b>5</b> | Limit, Continuity and Differentiability | 99 | 197 |
| <b>Mathematics</b>                      | <b>6</b> | Integral Calculus                       | 51 | 290 |
| Sets, Relations and Functions           | 35       | Differential Equations                  | 32 | 338 |
| Complex Numbers and Quadratic Equations | 48       | Coordinate Geometry                     | 94 | 372 |
| Matrices and Determinants               | 53       | Three Dimensional Geometry              | 39 | 473 |
| Permutations and Combinations           | 22       | Vector Algebra                          | 41 | 515 |
| Binomial Theorem and its Applications   | 9        | Statistics and Probability              | 52 | 555 |
| Sequence and Series                     | 31       | Trigonometry                            | 56 | 603 |
|                                         |          | Other Mathematics questions             | 3  | 659 |

# JEE Advanced 2007–2025

665 questions

# Mathematics

| Chapter                                 | All years | Last 5 years |
|-----------------------------------------|-----------|--------------|
| Sets, Relations and Functions           | 35        | 13           |
| Complex Numbers and Quadratic Equations | 48        | 10           |
| Matrices and Determinants               | 53        | 14           |
| Permutations and Combinations           | 22        | 4            |
| Binomial Theorem and its Applications   | 9         | 2            |
| Sequence and Series                     | 31        | 6            |
| Limit, Continuity and Differentiability | 99        | 15           |
| Integral Calculus                       | 51        | 11           |
| Differential Equations                  | 32        | 7            |
| Coordinate Geometry                     | 94        | 21           |
| Three Dimensional Geometry              | 39        | 8            |
| Vector Algebra                          | 41        | 10           |
| Statistics and Probability              | 52        | 20           |
| Trigonometry                            | 56        | 16           |
| Other Mathematics questions             | 3         | 0            |

## MATHEMATICS · CHAPTER 1

## Sets, Relations and Functions

35 questions · 2025 3 · 2024 6 · 2023 2 · 2022 2 · 2020 3 · 2019 1 · 2018 2 · 2014 3 · 2013 1 · 2012 2 · 2011 4 · 2010 2  
2009 1 · 2008 1 · 2007 2

## 1 JEE ADV 2025 · PAPER 1 · +3, -1

Let  $\mathbb{R}$  denote the set of all real numbers. Let  $a_i, b_i \in \mathbb{R}$  for  $i \in \{1, 2, 3\}$ . Define the functions  $f : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : \mathbb{R} \rightarrow \mathbb{R}$ , and  $h : \mathbb{R} \rightarrow \mathbb{R}$  by

$$f(x) = a_1 + 10x + a_2x^2 + a_3x^3 + x^4$$

$$g(x) = b_1 + 3x + b_2x^2 + b_3x^3 + x^4$$

$$h(x) = f(x+1) - g(x+2)$$

If  $f(x) \neq g(x)$  for every  $x \in \mathbb{R}$ , then the coefficient of  $x^3$  in  $h(x)$  is

- (A) 8 (B) 2  
(C) -4 (D) -6

## 2 JEE ADV 2025 · PAPER 1 · NUMERICAL · +4, 0

Let the set of all relations  $R$  on the set  $\{a, b, c, d, e, f\}$ , such that  $R$  is reflexive and symmetric, and  $R$  contains exactly 10 elements, be denoted by  $S$ . Then the number of elements in  $S$  is \_\_\_\_\_.

## 3 JEE ADV 2025 · PAPER 1 · ONE OR MORE CORRECT · +4, PARTIAL +1, -2

Let  $\mathbb{N}$  denote the set of all natural numbers, and  $\mathbb{Z}$  denote the set of all integers. Consider the functions  $f : \mathbb{N} \rightarrow \mathbb{Z}$  and  $g : \mathbb{Z} \rightarrow \mathbb{N}$  defined by

$$f(n) = \begin{cases} \frac{n+1}{2}, & n \text{ is odd} \\ \frac{4-n}{2}, & n \text{ is even} \end{cases}$$

and

$$g(n) = \begin{cases} 3+2n, & n \geq 0 \\ -2n, & n < 0 \end{cases}$$

Define  $(g \circ f)(n) = g(f(n))$  for all  $n \in \mathbb{N}$ , and  $(f \circ g)(n) = f(g(n))$  for all  $n \in \mathbb{Z}$ . Then which of the following statements is (are) TRUE?

- (A)  $g \circ f$  is NOT one-one and  $g \circ f$  is NOT onto  
(B)  $f \circ g$  is NOT one-one but  $f \circ g$  is onto  
(C)  $g$  is one-one and  $g$  is onto  
(D)  $f$  is NOT one-one but  $f$  is onto

## 4 JEE ADV 2024 · PAPER 1 · ONE OR MORE CORRECT · +4, PARTIAL +1, -2

Let  $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$ ,  $T_1 = \{(-1 + \sqrt{2})^n : n \in \mathbb{Z}\}$ , and  $T_2 = \{(1 + \sqrt{2})^n : n \in \mathbb{N}\}$ . Then which of the following statements is(are) TRUE?

- (A)  $\mathbb{Z} \cup T_1 \cup T_2 \subset S$   
(B)  $T_1 \cap (0, \frac{1}{2024}) = \emptyset$ , where  $\emptyset$  denotes the empty set  
(C)  $T_2 \cap (2024, \infty) \neq \emptyset$   
(D) For any given  $a, b \in \mathbb{Z}$ ,  $\cos(\pi(a + b\sqrt{2})) + i \sin(\pi(a + b\sqrt{2})) \in \mathbb{Z}$  if and only if  $b = 0$ , where  $i = \sqrt{-1}$

## 5 JEE ADV 2024 · PAPER 1 · MATCHING · +3, -1

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be functions defined by

$$f(x) = \begin{cases} x|x| \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and

$$g(x) = \begin{cases} 1-2x, & 0 \leq x \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

Let  $a, b, c, d \in \mathbb{R}$ . Define the function  $h : \mathbb{R} \rightarrow \mathbb{R}$  by

$$h(x) = af(x) + b(g(x) + g(\frac{1}{2} - x)) + c(x - g(x)) + dg(x), \quad x \in \mathbb{R}$$

Match each entry in List-I to the correct entries in List-II.

| List-I                                          | List-II                                     |
|-------------------------------------------------|---------------------------------------------|
| (P) If $a = 0, b = 1, c = 0$ and $d = 0$ , then | (1) $h$ is one-one.                         |
| (Q) If $a = 1, b = 0, c = 0$ and $d = 0$ , then | (2) $h$ is onto.                            |
| (R) If $a = 0, b = 0, c = 1$ and $d = 0$ , then | (3) $h$ is differentiable on $\mathbb{R}$ . |
| (S) If $a = 0, b = 0, c = 0$ and $d = 1$ , then | (4) the range of $h$ is $[0, 1]$ .          |
|                                                 | (5) the range of $h$ is $\{0, 1\}$ .        |

The correct option is:

- (A)  $(P) \rightarrow (4), (Q) \rightarrow (3), (R) \rightarrow (1), (S) \rightarrow (2)$   
(B)  $(P) \rightarrow (5), (Q) \rightarrow (2), (R) \rightarrow (4), (S) \rightarrow (3)$   
(C)  $(P) \rightarrow (5), (Q) \rightarrow (3), (R) \rightarrow (2), (S) \rightarrow (4)$   
(D)  $(P) \rightarrow (4), (Q) \rightarrow (2), (R) \rightarrow (1), (S) \rightarrow (3)$

**6 JEE ADV 2024 · PAPER 2 · +3, -1**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function defined by  $f(x) = x^2 \sin\left(\frac{\pi}{x^2}\right)$  if  $x \neq 0$ , and  $f(x) = 0$  if  $x = 0$ . Then which of the following statements is TRUE?

- (A)  $f(x) = 0$  has infinitely many solutions in the interval  $\left[\frac{1}{10^{10}}, \infty\right)$ .  
 (B)  $f(x) = 0$  has no solutions in the interval  $\left[\frac{1}{\pi}, \infty\right)$ .  
 (C) The set of solutions of  $f(x) = 0$  in the interval  $\left(0, \frac{1}{10^{10}}\right)$  is finite.  
 (D)  $f(x) = 0$  has more than 25 solutions in the interval  $\left(\frac{1}{\pi^2}, \frac{1}{\pi}\right)$ .

**7 JEE ADV 2024 · PAPER 2 · NUMERICAL · +4, 0**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that  $f(x+y) = f(x) + f(y)$  for all  $x, y \in \mathbb{R}$ , and  $g : \mathbb{R} \rightarrow (0, \infty)$  be a function such that  $g(x+y) = g(x)g(y)$  for all  $x, y \in \mathbb{R}$ . If  $f\left(-\frac{3}{5}\right) = 12$  and  $g\left(-\frac{1}{3}\right) = 2$ , then the value of  $f\left(\frac{1}{4}\right) + g(-2) - 8$  is \_\_\_\_\_

**PARAGRAPH**

Let  $S = \{1, 2, 3, 4, 5, 6\}$  and  $X$  be the set of all relations  $R$  from  $S$  to  $S$  that satisfy both the following properties:

- $R$  has exactly 6 elements.
- For each  $(a, b) \in R$ , we have  $|a - b| \geq 2$ .

Let  $Y = \{R \in X : \text{The range of } R \text{ has exactly one element}\}$  and  $Z = \{R \in X : R \text{ is a function from } S \text{ to } S\}$ . Let  $n(A)$  denote the number of elements in a set  $A$ .

There are two questions based on PARAGRAPH "I", the question given below is one of them.

**8 JEE ADV 2024 · PAPER 2 · NUMERICAL · +4, 0**

Paragraph I:

Let  $S = \{1, 2, 3, 4, 5, 6\}$  and  $X$  be the set of all relations  $R$  from  $S$  to  $S$  that satisfy both the following properties:

- $R$  has exactly 6 elements.
- For each  $(a, b) \in R$ , we have  $|a - b| \geq 2$ .

Let  $Y = \{R \in X : \text{The range of } R \text{ has exactly one element}\}$  and  $Z = \{R \in X : R \text{ is a function from } S \text{ to } S\}$ . Let  $n(A)$  denote the number of elements in a set  $A$ .

Question: If  $n(X) = {}^m C_6$ , then the value of  $m$  is \_\_\_\_\_.

**9 JEE ADV 2024 · PAPER 2 · NUMERICAL · +4, 0**

If the value of  $n(Y) + n(Z)$  is  $k^2$ , then  $|k|$  is \_\_\_\_\_.

**10 JEE ADV 2023 · PAPER 1 · NUMERICAL · +3, 0**

Let  $\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  for  $x \in \mathbb{R}$ . Then the number of real solutions of the equation

$$\sqrt{1 + \cos(2x)} = \sqrt{2} \tan^{-1}(\tan x)$$

in the set

$$\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right) \cup \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

is equal to

**11 JEE ADV 2023 · PAPER 2 · +3, -1**

For any  $y \in \mathbb{R}$ , let  $\cot^{-1}(y) \in (0, \pi)$  and  $\tan^{-1}(y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ . Then the sum of all the solutions of the equation  $\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \cot^{-1}\left(\frac{9-y^2}{6y}\right) = \frac{2\pi}{3}$  for  $0 < |y| < 3$ , is equal to

- (A)  $2\sqrt{3} - 3$  (B)  $3 - 2\sqrt{3}$   
 (C)  $4\sqrt{3} - 6$  (D)  $6 - 4\sqrt{3}$

**12 JEE ADV 2022 · PAPER 1 · NUMERICAL · +3, 0**

Considering only the principal values of the inverse trigonometric functions, the value of

$$\frac{3}{2} \cos^{-1} \sqrt{\frac{2}{2+\pi^2}} + \frac{1}{4} \sin^{-1} \left( \frac{2\sqrt{2}\pi}{2+\pi^2} \right) + \tan^{-1} \left( \frac{\sqrt{2}}{\pi} \right)$$

is \_\_\_\_\_.

**13 JEE ADV 2022 · PAPER 1 · NUMERICAL · +3, 0**

Let  $\alpha$  be a positive real number. Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : (\alpha, \infty) \rightarrow \mathbb{R}$  be the functions defined by

$$f(x) = \sin\left(\frac{\pi x}{12}\right)$$

and

$$g(x) = \frac{2 \log_e(\sqrt{x} - \sqrt{\alpha})}{\log_e(e^{\sqrt{x}} - e^{\sqrt{\alpha}})}.$$

Then the value of  $\lim_{x \rightarrow \alpha^+} f(g(x))$  is \_\_\_\_\_.

**14 JEE ADV 2020 · PAPER 2 · +3, -1**

If the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is defined by  $f(x) = |x|(x - \sin x)$ , then which of the following statements is TRUE?

- (A)  $f$  is one-one, but NOT onto (B)  $f$  is onto, but NOT one-one  
 (C)  $f$  is BOTH one-one and onto (D)  $f$  is NEITHER one-one NOR onto

**15 JEE ADV 2020 · PAPER 2 · +3, -1**

Let the functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = e^{x-1} - e^{-|x-1|}$$

and

$$g(x) = \frac{1}{2}(e^{x-1} + e^{1-x}).$$

Then the area of the region in the first quadrant bounded by the curves  $y = f(x)$ ,  $y = g(x)$  and  $x = 0$  is

(A)

$$(2 - \sqrt{3}) + \frac{1}{2}(e - e^{-1})$$

(B)

$$(2 + \sqrt{3}) + \frac{1}{2}(e - e^{-1})$$

(C)

$$(2 - \sqrt{3}) + \frac{1}{2}(e + e^{-1})$$

(D)

$$(2 + \sqrt{3}) + \frac{1}{2}(e + e^{-1})$$

**16 JEE ADV 2020 · PAPER 1 · NUMERICAL · +4, 0**

Let the function  $f : [0, 1] \rightarrow \mathbb{R}$  be defined by

$$f(x) = \frac{4^x}{4^x + 2}$$

Then the value of

$$f\left(\frac{1}{40}\right) + f\left(\frac{2}{40}\right) + f\left(\frac{3}{40}\right) + \cdots + f\left(\frac{39}{40}\right) - f\left(\frac{1}{2}\right)$$

is \_\_\_\_\_.

**17 JEE ADV 2019 · PAPER 2 · ONE OR MORE CORRECT · +4, PARTIAL +1, -1**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function. We say that  $f$  has PROPERTY 1 if

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{\sqrt{|h|}}$$

exists and is finite, and PROPERTY 2 if

$$\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h^2}$$

exists and is finite. Then which of the following options is/are correct?

(A)  $f(x) = x|x|$  has  
PROPERTY 2

(B)  $f(x) = \sin x$  has  
PROPERTY 2

(C)  $f(x) = |x|$  has  
PROPERTY 1

(D)  $f(x) = x^{2/3}$  has  
PROPERTY 1

**18 JEE ADV 2018 · PAPER 1 · NUMERICAL · +3, 0**

The number of real solutions of the equation

$$\sin^{-1} \left( \sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^i \right) = \frac{\pi}{2} -$$

$$\cos^{-1} \left( \sum_{i=1}^{\infty} \left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty} (-x)^i \right) \text{ lying in the interval}$$

$\left(-\frac{1}{2}, \frac{1}{2}\right)$  is \_\_\_\_\_. Here, the inverse trigonometric functions  $\sin^{-1} x$  and  $\cos^{-1} x$  assume values in  $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$  and  $[0, \pi]$ , respectively.

**19 JEE ADV 2018 · PAPER 2 · MATCHING · +3, -1**

Let  $E_1 = \{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\}$  and  $E_2 = \{x \in E_1 : \sin^{-1}(\log_e(\frac{x}{x-1})) \text{ is a real number}\}$ . Here, the inverse trigonometric function  $\sin^{-1} x$  assumes values in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ . Let  $f : E_1 \rightarrow \mathbb{R}$  be the function defined by  $f(x) = \log_e(\frac{x}{x-1})$  and  $g : E_2 \rightarrow \mathbb{R}$  be the function defined by  $g(x) = \sin^{-1}(\log_e(\frac{x}{x-1}))$ . Match LIST-I with LIST-II.

| LIST-I                        | LIST-II                                                    |
|-------------------------------|------------------------------------------------------------|
| P. The range of $f$ is        | 1. $(-\infty, \frac{1}{1-e}] \cup [\frac{e}{e-1}, \infty)$ |
| Q. The range of $g$ contains  | 2. $(0, 1)$                                                |
| R. The domain of $f$ contains | 3. $[-\frac{1}{2}, \frac{1}{2}]$                           |
| S. The domain of $g$ is       | 4. $(-\infty, 0) \cup (0, \infty)$                         |
|                               | 5. $(-\infty, \frac{e}{e-1}]$                              |
|                               | 6. $(-\infty, 0) \cup (\frac{1}{2}, \frac{e}{e-1}]$        |

The correct option is:

Let  $E_1 = \{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\}$   
 and  $E_2 = \{x \in E_1 : \sin^{-1}(\log_e(\frac{x}{x-1})) \text{ is a real number}\}$   
 (Here, the inverse trigonometric function  $\sin^{-1} x$  assumes values in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ )  
 Let  $f : E_1 \rightarrow \mathbb{R}$  be the function defined by  $f(x) = \log_e(\frac{x}{x-1})$   
 and  $g : E_2 \rightarrow \mathbb{R}$  be the function defined by  $g(x) = \sin^{-1}(\log_e(\frac{x}{x-1}))$ .

| LIST-I                        | LIST-II                                                    |
|-------------------------------|------------------------------------------------------------|
| P. The range of $f$ is        | 1. $(-\infty, \frac{1}{1-e}] \cup [\frac{e}{e-1}, \infty)$ |
| Q. The range of $g$ contains  | 2. $(0, 1)$                                                |
| R. The domain of $f$ contains | 3. $[-\frac{1}{2}, \frac{1}{2}]$                           |
| S. The domain of $g$ is       | 4. $(-\infty, 0) \cup (0, \infty)$                         |
|                               | 5. $(-\infty, \frac{e}{e-1}]$                              |
|                               | 6. $(-\infty, 0) \cup (\frac{1}{2}, \frac{e}{e-1}]$        |

(A)  $P \rightarrow 4; Q \rightarrow 2; R \rightarrow 1; S \rightarrow 1$

(B)  $P \rightarrow 3; Q \rightarrow 3; R \rightarrow 6; S \rightarrow 5$

(C)  $P \rightarrow 4; Q \rightarrow 2; R \rightarrow 1; S \rightarrow 6$

(D)  $P \rightarrow 4; Q \rightarrow 3; R \rightarrow 6; S \rightarrow 5$

**20 JEE ADV 2014 · PAPER 1 · ONE OR MORE CORRECT · +3, -1**

Let  $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$  be given by  $f(x) = (\log(\sec x + \tan x))^3$ . Then which of the following statements are correct?

(A)  $f(x)$  is an odd function (B)  $f(x)$  is a one-one function

(C)  $f(x)$  is an onto function (D)  $f(x)$  is an even function

**21 JEE ADV 2014 · PAPER 1 · ONE OR MORE CORRECT**

Let  $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$  be given by

$$f(x) = (\log(\sec x + \tan x))^3.$$

Then

(A)  $f(x)$  is an odd function (B)  $f(x)$  is a one-one function

(C)  $f(x)$  is an onto function (D)  $f(x)$  is an even function

**PARAGRAPH**

Paragraph for Questions 53 and 54

Given that for each  $a \in (0, 1)$ ,

$$\lim_{h \rightarrow 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1} dt$$

exists. Let this limit be  $g(a)$ . In addition, it is given that the function  $g(a)$  is differentiable on  $(0, 1)$ .

**22 JEE ADV 2014 · PAPER 2 · +3, -1**

The value of  $g(\frac{1}{2})$  is

(A)  $\pi$

(B)  $2\pi$

(C)

(D)

$$\frac{\pi}{2}$$

$$\frac{\pi}{4}$$

**23 JEE ADV 2013 · PAPER 2 · ONE OR MORE CORRECT · +4, -1**

The function  $f(x) = 2|x| + |x+2| - ||x+2| - 2|x||$  has a local minimum or a local maximum at  $x =$

(A)  $-2$

(B)

$$-\frac{2}{3}$$

(C) 2

(D)

$$\frac{2}{3}$$

**24 JEE ADV 2012 · PAPER 1 · +3, -1**

The function  $f : [0, 3] \rightarrow [1, 29]$ , defined by  $f(x) = 2x^3 - 15x^2 + 36x + 1$ , is

(A) one-one and onto

(B) onto but not one-one

(C) one-one but not onto

(D) neither one-one nor onto

**25 JEE ADV 2012 · PAPER 2 · +3, -1**

The value of the integral

$$\int_{-\pi/2}^{\pi/2} (x^2 + \ln \frac{\pi+x}{\pi-x}) \cos x dx$$

is

(A) 0

(B)

$$\frac{\pi^2}{2} - 4$$

(C)

(D)

$$\frac{\pi^2}{2} + 4$$

$$\frac{\pi^2}{2}$$

**26 JEE ADV 2011 · PAPER 1 · ONE OR MORE CORRECT · +4, -1**

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a function such that

$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}.$$

If  $f(x)$  is differentiable at  $x = 0$ , then which of the following statements is/are correct?

- (A)  $f(x)$  is differentiable only in a finite interval containing zero  
 (B)  $f(x)$  is continuous for all  $x \in \mathbb{R}$   
 (C)  $f'(x)$  is constant for all  $x \in \mathbb{R}$   
 (D)  $f(x)$  is differentiable except at finitely many points

**27 JEE ADV 2011 · PAPER 2 · +3, -1**

Let  $f(x) = x^2$  and  $g(x) = \sin x$  for all  $x \in \mathbb{R}$ . Then the set of all  $x$  satisfying  $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$ , where  $(f \circ g)(x) = f(g(x))$ , is

- (A)  $\pm\sqrt{n\pi}$ ,  $n \in \{0, 1, 2, \dots\}$   
 (B)  $\pm\sqrt{n\pi}$ ,  $n \in \{1, 2, \dots\}$   
 (C)

$$\frac{\pi}{2} + 2n\pi, n \in \{\dots, -2, -1, 0, 1, 2, \dots\}$$

- (D)  $2n\pi$ ,  $n \in \{\dots, -2, -1, 0, 1, 2, \dots\}$

**28 JEE ADV 2011 · PAPER 2 · ONE OR MORE CORRECT · +4, -1**

Let  $f : (0, 1) \rightarrow \mathbb{R}$  be defined by

$$f(x) = \frac{b-x}{1-bx}$$

where  $b$  is a constant such that  $0 < b < 1$ . Then choose the correct option(s).

- (A)  $f$  is not invertible on  $(0, 1)$   
 (B)  $f \neq f^{-1}$  on  $(0, 1)$  and  $f'(b) = \frac{1}{f'(0)}$   
 (C)  $f = f^{-1}$  on  $(0, 1)$  and  $f'(b) = \frac{1}{f'(0)}$   
 (D)  $f^{-1}$  is differentiable on  $(0, 1)$

**29 JEE ADV 2011 · PAPER 2 · MATCHING**

Match the statements given in Column I with the values given in Column II.

| Column I                                                                                                                                                                                                           | Column II            |
|--------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------|
| (A) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}$ , $\vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is | (p) $\frac{\pi}{6}$  |
| (B) If $\int_a^b (f(x) - 3x) dx = a^2 - b^2$ , then the value of $f\left(\frac{\pi}{6}\right)$ is                                                                                                                  | (q) $\frac{2\pi}{3}$ |
| (C) The value of $\frac{\pi^2}{\ln 3} \int_{-1/6}^{1/6} \sec(\pi x) dx$ is                                                                                                                                         | (r) $\frac{\pi}{3}$  |
| (D) The maximum value of $ \operatorname{Arg}\left(\frac{1}{1-z}\right) $ for $ z  = 1$ , $z \neq 1$ , is given by                                                                                                 | (s) $\pi$            |
|                                                                                                                                                                                                                    | (t) $\frac{\pi}{2}$  |

(p)

$$\frac{\pi}{6}$$

(q)

$$\frac{2\pi}{3}$$

(r)

$$\frac{\pi}{3}$$

(s)  $\pi$ 

(t)

$$\frac{\pi}{2}$$

**30 JEE ADV 2010 · PAPER 2 · +3, -1**

Let  $f$  be a real-valued function defined on the interval  $(-1, 1)$  such that

$$e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4 + 1} dt$$

for all  $x \in (-1, 1)$ , and let  $f^{-1}$  be the inverse function of  $f$ . Then  $(f^{-1})'(2)$  is equal to

(A) 1

(B)

$$\frac{1}{3}$$

(C)

$$\frac{1}{2}$$

(D)

$$\frac{1}{e}$$

**31 JEE ADV 2010 · PAPER 2 · +3, -1**

Let  $S = \{1, 2, 3, 4\}$ . The total number of unordered pairs of disjoint subsets of  $S$  is equal to

(A) 25

(B) 34

(C) 42

(D) 41

**32 JEE ADV 2009 · PAPER 2 · ONE OR MORE CORRECT · +4, -1**

For the function  $f(x) = x \cos \frac{1}{x}$ ,  $x \geq 1$ , which of the following statements are correct?

- (A) For at least one  $x$  in the interval  $[1, \infty)$ ,  $f(x+2) - f(x) < 2$   
 (B)

$$\lim_{x \rightarrow \infty} f'(x) = 1$$

- (C) For all  $x$  in the interval  $[1, \infty)$ ,  $f(x+2) - f(x) > 2$   
 (D)  $f'(x)$  is strictly decreasing in the interval  $[1, \infty)$

**33 JEE ADV 2008 · PAPER 2 · +3, -1**

Let the function  $g : (-\infty, \infty) \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$  be given by  $g(u) = 2 \tan^{-1}(e^u) - \frac{\pi}{2}$ . Then,  $g$  is

- (A) even and is strictly increasing in  $(0, \infty)$   
 (B) odd and is strictly decreasing in  $(-\infty, \infty)$   
 (C) odd and is strictly increasing in  $(-\infty, \infty)$   
 (D) neither even nor odd, but is strictly increasing in  $(-\infty, \infty)$

**34 JEE ADV 2007 · PAPER 2**

The line  $y = x$  meets  $y = ke^x$  for  $k \leq 0$  at

- (A) no point (B) one point  
 (C) two points (D) more than two points

**35 JEE ADV 2007 · PAPER 2 · MATCHING · +6, -2**

Let  $(x, y)$  be such that

$$\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$$

Match the statements in Column I with statements in Column II and indicate your answer by darkening the appropriate bubbles in the  $4 \times 4$  matrix given in the ORS.

| Column I                                   | Column II                              |
|--------------------------------------------|----------------------------------------|
| (A) If $a = 1$ and $b = 0$ , then $(x, y)$ | (p) lies on the circle $x^2 + y^2 = 1$ |
| (B) If $a = 1$ and $b = 1$ , then $(x, y)$ | (q) lies on $(x^2 - 1)(y^2 - 1) = 0$   |
| (C) If $a = 1$ and $b = 2$ , then $(x, y)$ | (r) lies on $y = x$                    |
| (D) If $a = 2$ and $b = 2$ , then $(x, y)$ | (s) lies on $(4x^2 - 1)(y^2 - 1) = 0$  |

**Answer key · Sets, Relations and Functions**

|        |            |            |           |                       |             |
|--------|------------|------------|-----------|-----------------------|-------------|
| 1 C    | 2 105      | 3 A, D     | 4 A, C, D | 5 C                   | 6 D         |
| 7 51   | 8 20.00    | 9 36.00    | 10 3      | 11 C                  | 12 $3\pi/4$ |
| 13 0.5 | 14 C       | 15 A       | 16 19.00  | 17 C, D               | 18 2        |
| 19 A   | 20 A, B, C | 21 A, B, C | 22 A      | 23 A, B               | 24 B        |
| 25 B   | 26 B, C    | 27 A       | 28 A      | 29 p, q, s, t         | 30 B        |
| 31 D   | 32 B, C, D | 33 C       | 34 B      | 35 A-p, B-q, C-p, D-s |             |

**Solutions · Sets, Relations and Functions****1 Answer: C**

Ref 14936

Hard ~3 min Polynomial equations and real roots

**QUICK APPROACH** Start by expanding  $f(x+1)$  and  $g(x+2)$  using the Binomial theorem to find the coefficient of  $x^3$  in each; then use the condition that  $f(x) \neq g(x)$  for all real  $x$  to relate the coefficients.

We need the coefficient of  $x^3$  in

$$h(x) = f(x+1) - g(x+2)$$

First find the coefficient of  $x^3$  in  $f(x+1)$ .

$$f(x+1) = a_1 + 10(x+1) + a_2(x+1)^2 + a_3(x+1)^3 + (x+1)^4$$

Only  $a_3(x+1)^3$  and  $(x+1)^4$  contribute to the coefficient of  $x^3$ .

$$(x+1)^3 = x^3 + 3x^2 + 3x + 1$$

So the coefficient of  $x^3$  from  $a_3(x+1)^3$  is  $a_3$ .

$$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$$

So the coefficient of  $x^3$  from  $(x+1)^4$  is 4.

Therefore, the coefficient of  $x^3$  in  $f(x+1)$  is

$$a_3 + 4$$

Now find the coefficient of  $x^3$  in  $g(x+2)$ .

$$g(x+2) = b_1 + 3(x+2) + b_2(x+2)^2 + b_3(x+2)^3 + (x+2)^4$$

Only  $b_3(x+2)^3$  and  $(x+2)^4$  contribute to the coefficient of  $x^3$ .

$$(x+2)^3 = x^3 + 6x^2 + 12x + 8$$

So the coefficient of  $x^3$  from  $b_3(x+2)^3$  is  $b_3$ .

$$(x+2)^4 = x^4 + 8x^3 + 24x^2 + 32x + 16$$

So the coefficient of  $x^3$  from  $(x+2)^4$  is 8.

Therefore, the coefficient of  $x^3$  in  $g(x+2)$  is

$$b_3 + 8$$

Hence, the coefficient of  $x^3$  in  $h(x)$  is

$$\begin{aligned}(a_3 + 4) - (b_3 + 8) \\ = a_3 - b_3 - 4\end{aligned}$$

Now use the condition  $f(x) \neq g(x)$  for every real  $x$ .

This means

$$f(x) - g(x) \neq 0$$

for every real  $x$ .

Compute  $f(x) - g(x)$ :

$$f(x) - g(x) = (a_1 - b_1) + 7x + (a_2 - b_2)x^2 + (a_3 - b_3)x^3$$

If  $a_3 - b_3 \neq 0$ , then  $f(x) - g(x)$  is a cubic polynomial with real coefficients. Every real cubic polynomial has at least one real root. That would contradict the condition that  $f(x) - g(x) \neq 0$  for every real  $x$ .

Therefore,

$$a_3 - b_3 = 0$$

So the required coefficient is

$$\begin{aligned}a_3 - b_3 - 4 \\ = 0 - 4 \\ = -4\end{aligned}$$

Thus, the correct option is C.

#### COMMON TRAPS

- **Using the condition on shifted functions instead of original functions.** The exact condition is for the same input  $x$  in  $f$  and  $g$ :  $f(x) - g(x) \neq 0$  for every real  $x$ . It is not a statement about  $f(x+1) - g(x+2)$ . *Avoid it:* Before using a condition, rewrite it literally with the same variables:  $f(t) - g(t) \neq 0$  for all real  $t$ , then compute  $f(t) - g(t)$ .
- **Not cancelling the  $x^4$  terms in  $f(x) - g(x)$ .** Since both polynomials contain the exact same term  $+x^4$ ,  $f(x) - g(x)$  has no  $x^4$  term. *Avoid it:* Line up like powers when subtracting:  $(x^4) - (x^4) = 0$ , then compare cubic, quadratic, linear, constant terms.

**REVISE** Practise binomial coefficient extraction without full expansion, especially coefficients of  $x^{n-1}$  in  $(x+c)^n$ . The real-root property of odd-degree polynomials and how it differs from quadratic discriminant arguments.

2 Answer: 105

Ref 14938

Medium ~2 min Counting reflexive and symmetric relations

Let the given set be  $A = \{a, b, c, d, e, f\}$ .

The number of elements in  $A$  is

$$|A| = 6$$

A relation  $R$  on  $A$  is a subset of  $A \times A$ .

Since  $R$  is reflexive, all ordered pairs of the form  $(x, x)$  must belong to  $R$ .

So the compulsory reflexive pairs are

$$(a, a), (b, b), (c, c), (d, d), (e, e), (f, f)$$

Hence, 6 elements are already fixed in  $R$ .

The relation contains exactly 10 elements. Therefore, the number of additional elements required is

$$10 - 6 = 4$$

Now  $R$  is symmetric. So whenever  $(x, y) \in R$  with  $x \neq y$ , we must also have  $(y, x) \in R$ .

Thus, non-diagonal elements must be selected in pairs:

$$(x, y) \text{ and } (y, x)$$

Since we need 4 additional ordered pairs, we must choose 2 unordered pairs from the 6 elements of  $A$ .

The total number of unordered pairs from 6 elements is

$$\binom{6}{2} = 15$$

We need to choose 2 such unordered pairs. Therefore, the number of possible relations is

$$\binom{15}{2} = \frac{15 \times 14}{2}$$

$$\binom{15}{2} = 105$$

Hence, the number of elements in  $S$  is

$$105$$

3 Answer: A, D

Ref 14942

Hard ~4 min

First analyze  $f$ .

For odd natural numbers, write

$$n = 2k + 1$$

where  $k \in \mathbb{N} \cup \{0\}$ .

Then

$$f(n) = \frac{n+1}{2}$$

$$f(2k+1) = \frac{2k+2}{2}$$

$$f(2k+1) = k+1$$

So odd inputs give all positive integers  $1, 2, 3, \dots$

For even natural numbers, write

$$n = 2k$$

where  $k \in \mathbb{N}$ .

Then

$$f(n) = \frac{4-n}{2}$$

$$f(2k) = \frac{4-2k}{2}$$

$$f(2k) = 2-k$$

As  $k = 1, 2, 3, \dots$ , the values are

$$1, 0, -1, -2, \dots$$

Thus the range of  $f$  contains all positive integers from odd inputs and all non-positive integers from even inputs. Hence

$$\text{Range}(f) = \mathbb{Z}$$

So  $f$  is onto.

But

$$f(1) = \frac{1+1}{2}$$

$$f(1) = 1$$

and

$$f(2) = \frac{4-2}{2}$$

$$f(2) = 1$$

Since two different inputs have the same image,  $f$  is not one-one. Therefore option D is true.

Now analyze  $g$ .

For  $n \geq 0$ ,

$$g(n) = 3 + 2n$$

This gives odd natural numbers  $3, 5, 7, \dots$

For  $n < 0$ , write  $n = -m$ , where  $m \in \mathbb{N}$ .

Then

$$g(n) = -2n$$

$$g(-m) = 2m$$

This gives even natural numbers  $2, 4, 6, \dots$

Therefore

$$\text{Range}(g) = \mathbb{N} \setminus \{1\}$$

So  $g$  is not onto. Hence option C is false.

Also, the two branches of  $g$  give disjoint types of values: odd values from  $n \geq 0$  and even values from  $n < 0$ . Each branch is individually one-one, so  $g$  is one-one but not onto.

Now analyze  $g \circ f$ .

We already know

$$f(1) = 1$$

$$f(2) = 1$$

So

$$(g \circ f)(1) = g(f(1))$$

$$(g \circ f)(1) = g(1)$$

$$(g \circ f)(1) = 5$$

and

$$(g \circ f)(2) = g(f(2))$$

$$(g \circ f)(2) = g(1)$$

$$(g \circ f)(2) = 5$$

Thus  $g \circ f$  is not one-one.

To check onto, observe that  $g$  never gives the value 1. Since  $g \circ f$  always outputs a value of  $g$ , it also can never give 1.

So  $g \circ f$  is not onto. Therefore option A is true.

Now analyze  $f \circ g$ .

If  $n \geq 0$ , then

$$g(n) = 3 + 2n$$

This is odd. Hence

$$f(g(n)) = \frac{g(n) + 1}{2}$$

$$f(g(n)) = \frac{3 + 2n + 1}{2}$$

$$f(g(n)) = n + 2$$

If  $n < 0$ , then

$$g(n) = -2n$$

This is even. Hence

$$f(g(n)) = \frac{4 - g(n)}{2}$$

$$f(g(n)) = \frac{4 + 2n}{2}$$

$$f(g(n)) = n + 2$$

Therefore, for every integer  $n$ ,

$$(f \circ g)(n) = n + 2$$

The function  $n \mapsto n + 2$  from  $\mathbb{Z}$  to  $\mathbb{Z}$  is one-one and onto.  
Hence option B is false because it says  $f \circ g$  is not one-one.

Therefore the correct options are A and D.

#### COMMON TRAPS

- **Thinking  $f$  is one-one because odd and even branches look different.** A function is injective only if no two distinct inputs anywhere in the whole domain have the same output. *Avoid it:* After checking each branch, always compare boundary/simple values from different branches.
- **Assuming  $g$  is onto  $\mathbb{N}$  because it gives positive values.** Onto  $\mathbb{N}$  means every natural number, including 1, must be hit; here range is  $\{3, 5, 7, \dots\} \cup \{2, 4, 6, \dots\} = \mathbb{N} \setminus \{1\}$ . *Avoid it:* For onto questions, explicitly test the smallest codomain values 1, 2, 3 before declaring onto.

**REVIEW** Make small value tables for piecewise functions before proving injectivity or surjectivity. For onto checks into  $\mathbb{N}$  or  $\mathbb{Z}$ , explicitly test missing small values and solve  $h(x)=y$ .

#### 4 Answer: A, C, D

Ref 14838

**Hard** ~3 min Sets involving surds and complex exponential

Let  $\alpha = \sqrt{2} - 1$  and  $\beta = \sqrt{2} + 1$ . Then  $T_1 = \{\alpha^n : n \in \mathbb{Z}\}$  and  $T_2 = \{\beta^n : n \in \mathbb{N}\}$ .

For option A: First, every integer  $m \in \mathbb{Z}$  can be written as  $m + 0\sqrt{2}$ , so  $\mathbb{Z} \subset S$ .

Also,  $\alpha = \sqrt{2} - 1 = -1 + 1\sqrt{2} \in S$  and  $\beta = \sqrt{2} + 1 = 1 + 1\sqrt{2} \in S$ .

The set  $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$  is closed under multiplication because

$$(a + b\sqrt{2})(c + d\sqrt{2}) = (ac + 2bd) + (ad + bc)\sqrt{2}$$

where  $ac + 2bd$  and  $ad + bc$  are integers.

Hence all positive powers of  $\alpha$  and  $\beta$  lie in  $S$ . For negative powers of  $\alpha$ , note that

$$\alpha^{-1} = \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1 = \beta \in S$$

So  $\alpha^n \in S$  for every  $n \in \mathbb{Z}$ . Therefore  $T_1 \subset S$  and  $T_2 \subset S$ .  
Hence option A is true.

For option B: Since

$$0 < \alpha = \sqrt{2} - 1 < 1$$

we have

$$\alpha^n \rightarrow 0$$

as  $n \rightarrow \infty$ . Therefore for sufficiently large positive integer  $n$ ,

$$0 < \alpha^n < \frac{1}{2024}$$

Thus  $T_1 \cap (0, \frac{1}{2024})$  is not empty. Hence option B is false.

For option C: Since

$$\beta = 1 + \sqrt{2} > 1$$

its powers grow without bound. Therefore for sufficiently large  $n \in \mathbb{N}$ ,

$$\beta^n > 2024$$

So  $T_2 \cap (2024, \infty) \neq \emptyset$ . Hence option C is true.

For option D: The given complex number is

$$\cos(\pi(a + b\sqrt{2})) + i \sin(\pi(a + b\sqrt{2})) = e^{i\pi(a + b\sqrt{2})}$$

This number has modulus 1. If it is an integer, the only possible integer values are 1 and  $-1$ . This happens exactly when

$$e^{i\pi(a + b\sqrt{2})} = \pm 1$$

which is equivalent to

$$a + b\sqrt{2} \in \mathbb{Z}$$

Since  $a \in \mathbb{Z}$  and  $\sqrt{2}$  is irrational,  $a + b\sqrt{2}$  is an integer if and

only if  $b = 0$ .

If  $b = 0$ , then

$$e^{i\pi a} = (-1)^a \in \mathbb{Z}$$

Hence option D is true.

Therefore, the correct options are A, C and D.

#### COMMON TRAPS

- **Treating negative powers in T1 as automatically outside S.** Here  $\alpha = \sqrt{2} - 1$  has reciprocal  $\alpha^{-1} = \frac{1}{\sqrt{2}-1} = \sqrt{2} + 1$ , which is still in S. Hence every negative power of  $\alpha$  is a positive power of  $\sqrt{2} + 1$  and remains in S. *Avoid it:* When a set uses  $n \in \mathbb{Z}$ , separately check  $n > 0$ ,  $n = 0$ , and  $n < 0$ . For  $n < 0$ , compute the base reciprocal explicitly before deciding membership.
- **Thinking  $-1 + \sqrt{2}$  is negative.**  $-1 + \sqrt{2} = \sqrt{2} - 1 \approx 0.414$ , so it is positive and less than 1. Therefore positive powers stay positive and tend to 0. *Avoid it:* Before analysing powers, estimate the base numerically:  $\sqrt{2} - 1 \approx 0.414$ , not negative.

**REVISE** Closure of  $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$  under multiplication and special reciprocals like  $(\sqrt{2} - 1)^{-1} = \sqrt{2} + 1$ . Practice power behaviour for bases in  $(0, 1)$  versus bases greater than 1, especially with integer exponent sets.

#### 5 Answer: C

Ref 14851

Hard ~4 min Piecewise functions, range, onto and differentiability

First evaluate the auxiliary expression involving  $g$ .

For  $0 \leq x \leq \frac{1}{2}$ ,

$$g(x) = 1 - 2x$$

Also,

$$0 \leq \frac{1}{2} - x \leq \frac{1}{2}$$

so

$$g\left(\frac{1}{2} - x\right) = 1 - 2\left(\frac{1}{2} - x\right)$$

$$g\left(\frac{1}{2} - x\right) = 2x$$

Hence, for  $0 \leq x \leq \frac{1}{2}$ ,

$$g(x) + g\left(\frac{1}{2} - x\right) = 1$$

For  $x < 0$ , both  $g(x)$  and  $g\left(\frac{1}{2} - x\right)$  are 0. For  $x > \frac{1}{2}$ , both are again 0. Therefore,

$$g(x) + g\left(\frac{1}{2} - x\right) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2} \\ 0, & x \notin [0, \frac{1}{2}] \end{cases}$$

Now check each case.

For (P),  $a = 0, b = 1, c = 0, d = 0$ . Thus,

$$h(x) = g(x) + g\left(\frac{1}{2} - x\right)$$

So,

$$h(x) = \begin{cases} 1, & 0 \leq x \leq \frac{1}{2} \\ 0, & x \notin [0, \frac{1}{2}] \end{cases}$$

Hence the range of  $h$  is exactly  $\{0, 1\}$ . Therefore,  $(P) \rightarrow (5)$ .

For (Q),  $a = 1, b = 0, c = 0, d = 0$ . Thus,

$$h(x) = f(x)$$

For  $x \neq 0$ ,  $f(x) = x|x| \sin\left(\frac{1}{x}\right)$  is differentiable as a product and composition of differentiable functions. At  $x = 0$ , check differentiability from the definition.

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$$

$$f'(0) = \lim_{x \rightarrow 0} \frac{x|x| \sin\left(\frac{1}{x}\right)}{x}$$

$$f'(0) = \lim_{x \rightarrow 0} |x| \sin\left(\frac{1}{x}\right)$$

Since

$$-|x| \leq |x| \sin\left(\frac{1}{x}\right) \leq |x|$$

by squeeze theorem,

$$f'(0) = 0$$

Therefore,  $h$  is differentiable on  $\mathbb{R}$ . Hence,  $(Q) \rightarrow (3)$ .

For (R),  $a = 0, b = 0, c = 1, d = 0$ . Thus,

$$h(x) = x - g(x)$$

Using the definition of  $g$ ,

$$h(x) = \begin{cases} x, & x < 0 \\ x - (1 - 2x), & 0 \leq x \leq \frac{1}{2} \\ x, & x > \frac{1}{2} \end{cases}$$

So,

$$h(x) = \begin{cases} x, & x < 0 \\ 3x - 1, & 0 \leq x \leq \frac{1}{2} \\ x, & x > \frac{1}{2} \end{cases}$$

The range for  $x < 0$  is  $(-\infty, 0)$ . The range for  $0 \leq x \leq \frac{1}{2}$  is  $[-1, \frac{1}{2}]$ . The range for  $x > \frac{1}{2}$  is  $(\frac{1}{2}, \infty)$ . Their union is

$$(-\infty, 0) \cup [-1, \frac{1}{2}] \cup (\frac{1}{2}, \infty) = \mathbb{R}$$

Hence  $h$  is onto. Therefore,  $(R) \rightarrow (2)$ .

For (S),  $a = 0, b = 0, c = 0, d = 1$ . Thus,

$$h(x) = g(x)$$

$$\text{On } 0 \leq x \leq \frac{1}{2},$$

$$g(x) = 1 - 2x$$

As  $x$  moves from 0 to  $\frac{1}{2}$ ,  $g(x)$  moves from 1 to 0. Outside this interval,  $g(x) = 0$ , which is already included. Hence the range of  $h$  is

$$[0, 1]$$

Therefore,  $(S) \rightarrow (4)$ .

Thus the correct matching is

$$(P) \rightarrow (5), (Q) \rightarrow (3), (R) \rightarrow (2), (S) \rightarrow (4)$$

So the correct option is C.

#### COMMON TRAPS

- **Treating the sum of two shifted g-functions as a ramp instead of an indicator.** Because both terms are nonzero only when  $0 \leq x \leq \frac{1}{2}$ , their sum is exactly 1 on that whole interval and 0 outside it. Thus the range is  $\{0, 1\}$ , not  $[0, 1]$ . *Avoid it:* Always write a separate piecewise form for shifted functions before discussing range.
- **Misreading the endpoint condition of  $g$  as open or excluding 0 and  $1/2$ .** At  $x = 0$ ,  $g(0) = 1$ . At  $x = \frac{1}{2}$ ,  $g(\frac{1}{2}) = 0$ . Outside also gives 0, so the full range is exactly  $[0, 1]$ . *Avoid it:* Evaluate endpoints explicitly before taking the range of a piecewise linear function.

**REVISE** Before matching options, substitute the given values of  $a, b, c, d$  and write the exact simplified  $h(x)$  for P, Q, R, and S. For any piecewise function, mark breakpoints and test endpoint values explicitly.

#### 6 Answer: D

Ref 14885

Medium ~2 min Zeros of a function involving trigonometric equation

For  $x \neq 0$ , the equation is  $x^2 \sin\left(\frac{\pi}{x^2}\right) = 0$ . Since  $x^2 \neq 0$ , we need  $\sin\left(\frac{\pi}{x^2}\right) = 0$ . This happens when  $\frac{\pi}{x^2} = n\pi$ , where  $n$  is a positive integer. Hence  $x^2 = \frac{1}{n}$  and  $x = \pm \frac{1}{\sqrt{n}}$ . Also, from the

definition,  $x = 0$  is a solution. Now check each option. For option A, in  $[\frac{1}{10^{10}}, \infty)$ , positive roots satisfy  $\frac{1}{\sqrt{n}} \geq \frac{1}{10^{10}}$ , so  $n \leq 10^{20}$ . This gives finitely many positive roots, and no negative roots lie in the interval. So option A is false. For option B,  $x = 1$  is a solution because it corresponds to  $n = 1$ , and  $1 \in [\frac{1}{\pi}, \infty)$ . So option B is false. For option C, in  $(0, \frac{1}{10^{10}})$ , roots satisfy  $0 < \frac{1}{\sqrt{n}} < \frac{1}{10^{10}}$ , so  $n > 10^{20}$ . There are infinitely many such positive integers  $n$ , so option C is false. For option D, roots in  $(\frac{1}{\pi^2}, \frac{1}{\pi})$  satisfy  $\frac{1}{\pi^2} < \frac{1}{\sqrt{n}} < \frac{1}{\pi}$ . Taking reciprocals carefully gives  $\pi < \sqrt{n} < \pi^2$ . Squaring gives  $\pi^2 < n < \pi^4$ . Since  $\pi^2 \approx 9.87$  and  $\pi^4 \approx 97.4$ , the possible integer values are  $n = 10, 11, \dots, 97$ . The number of such integers is  $97 - 10 + 1 = 88$ , which is more than 25. Therefore, the correct answer is D.

#### 7 Answer: 51

Ref 14891

Medium ~2 min Functional equations

**QUICK APPROACH** Use the properties of additive and multiplicative functions, specifically that  $f(x+y)=f(x)+f(y)$  implies  $f$  is linear for rationals, and  $g(x+y)=g(x)g(y)$  implies an exponential form for  $g$ .

For the additive function  $f$ , we use the property that for rational numbers  $q$ ,  $f(q) = qf(1)$ .

Given:

$$f\left(-\frac{3}{5}\right) = 12$$

So,

$$-\frac{3}{5}f(1) = 12$$

$$f(1) = -20$$

Now,

$$f\left(\frac{1}{4}\right) = \frac{1}{4}f(1)$$

$$f\left(\frac{1}{4}\right) = -5$$

For the function  $g$ , first find  $g(0)$ .

$$g(0) = g(0+0)$$

$$g(0) = g(0)^2$$

Since  $g(x) > 0$ , we get:

$$g(0) = 1$$

Given:

$$g\left(-\frac{1}{3}\right) = 2$$

Since  $-2 = 6\left(-\frac{1}{3}\right)$ , using  $g(x+y) = g(x)g(y)$  repeatedly:

$$g(-2) = \left(g\left(-\frac{1}{3}\right)\right)^6$$

$$g(-2) = 2^6$$

$$g(-2) = 64$$

Now substitute in the required expression:

$$\left(f\left(\frac{1}{4}\right) + g(-2) - 8\right)g(0)$$

$$= (-5 + 64 - 8)(1)$$

$$= 51$$

Therefore, the required value is 51.

**8 Answer: 20.00**

Ref 14896

Medium ~2 min Counting relations satisfying a condition

We need to count all possible relations  $R$  from  $S$  to  $S$  which have exactly 6 ordered pairs and satisfy  $|a - b| \geq 2$  for every ordered pair  $(a, b)$ .

A relation from  $S$  to  $S$  is any subset of  $S \times S$ . First count the number of allowed ordered pairs  $(a, b)$  such that  $|a - b| \geq 2$ .

The total number of ordered pairs in  $S \times S$  is:

$$6 \times 6 = 36$$

Now count the disallowed ordered pairs for which  $|a - b| < 2$ . Since  $a, b$  are integers, this means:

$$|a - b| = 0 \text{ or } |a - b| = 1$$

For  $|a - b| = 0$ , the pairs are  $(1, 1), (2, 2), \dots, (6, 6)$ , so there are:

$$6$$

For  $|a - b| = 1$ , the adjacent unordered pairs are  $(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)$ . Each gives two ordered pairs, so the number is:

$$5 \times 2 = 10$$

Therefore, the number of disallowed ordered pairs is:

$$6 + 10 = 16$$

Hence, the number of allowed ordered pairs is:

$$36 - 16 = 20$$

Now  $X$  is the set of all relations  $R$  that contain exactly 6 elements chosen from these 20 allowed ordered pairs. Therefore:

$$n(X) = {}^{20}_6$$

Given:

$$n(X) = {}^m C_6$$

So:

$${}^m C_6 = {}^{20} C_6$$

Thus:

$$m = 20$$

Final answer: 20.00

#### COMMON TRAPS

- **Counting unordered pairs instead of ordered pairs.** A relation from  $S$  to  $S$  is a subset of  $S \times S$ , whose elements are ordered pairs. If  $a \neq b$ ,  $(a, b)$  and  $(b, a)$  are two distinct possible relation elements. *Avoid it:* Whenever you see 'from  $S$  to  $S$ ' or ' $(a, b) \in R$ ', write 'ordered pairs' explicitly before counting.
- **Mistaking relation for function.**  $X$  contains all relations with 6 ordered pairs;  $Z$  is only a subset of  $X$  where  $R$  is a function. For  $n(X)$ , no function condition is imposed. *Avoid it:* Separate the definitions: solve only for  $X$  in this question; ignore  $Y$  and  $Z$  unless explicitly asked.

**REVIEW** The distinction between relation and function: relation = arbitrary subset of  $A \times B$ ; function = exactly one output for every input. · Practise counting ordered pairs by difference  $d = |a - b|$  for small sets like  $\{1, 2, \dots, n\}$ .

**9 Answer: 36.00**

Ref 14897

Medium ~2 min Counting relations using range restrictions

**QUICK APPROACH** List, for each element, how many elements are at distance at least 2 from it; a function must use exactly one of them for every  $a$ .

Count the relations whose range is one element, then the functions, using the allowed pairs with  $|a - b| \geq 2$ .

**Step 1.** For a fixed second element  $b$ , the allowed first elements  $a$  are:  $b = 1$ :  $\{3, 4, 5, 6\}$ ;  $b = 2$ :  $\{4, 5, 6\}$ ;  $b = 3$ :  $\{1, 5, 6\}$ ;  $b = 4$ :  $\{1, 2, 6\}$ ;  $b = 5$ :  $\{1, 2, 3\}$ ;  $b = 6$ :  $\{1, 2, 3, 4\}$ .

**Step 2.** A relation in  $Y$  needs 6 pairs with the same  $b$ , but no  $b$  has more than 4 allowed partners, so  $n(Y) = 0$ .

**Step 3.** A function in  $Z$  picks one image for each  $a$  with  $|a - f(a)| \geq 2$ :  $a = 1$  and  $a = 6$  have 4 choices each,  $a = 2, 3, 4, 5$  have 3 each. So  $n(Z) = 4 \cdot 3^4 \cdot 4 = 1296$ .

**Step 4.**  $k^2 = 0 + 1296$ , so  $|k| = 36$ .

Answer: 36

## COMMON TRAPS

- **Counting functions with all 6 images free.** Each  $a$  may only map to an element at distance at least 2: 4 choices for  $a = 1, 6$  and 3 for  $a = 2, \dots, 5$ . *Avoid it:* Write the allowed images of each element before multiplying.
- **Forgetting that a relation in  $Y$  must still have 6 pairs.** A single-element range  $\{b\}$  means all 6 pairs share  $b$ , which needs 6 allowed  $a$ ; none exists, so  $Y$  is empty. *Avoid it:* Check the size condition on  $R$  alongside the range condition.

**REVISE** Counting relations and functions on a finite set under a restriction.

10 Answer: 3

Ref 14739

Hard ~3 min

Given principal branch:

$$\tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Use the identity:

$$1 + \cos(2x) = 2 \cos^2 x$$

Therefore,

$$\sqrt{1 + \cos(2x)} = \sqrt{2 \cos^2 x}$$

$$\sqrt{1 + \cos(2x)} = \sqrt{2} |\cos x|$$

The equation becomes:

$$\sqrt{2} |\cos x| = \sqrt{2} \tan^{-1}(\tan x)$$

Dividing by  $\sqrt{2}$ :

$$|\cos x| = \tan^{-1}(\tan x)$$

Now evaluate  $\tan^{-1}(\tan x)$  on each interval.For  $x \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right)$ :

$$\tan^{-1}(\tan x) = x + \pi$$

So the equation is:

$$|\cos x| = x + \pi$$

For the right side to be non-negative, we need:

$$x + \pi \geq 0$$

$$x \geq -\pi$$

Hence consider  $x \in [-\pi, -\frac{\pi}{2})$ . Put:

$$u = x + \pi$$

Then:

$$u \in [0, \frac{\pi}{2})$$

Also,

$$|\cos x| = |\cos(u - \pi)|$$

$$|\cos x| = |-\cos u|$$

$$|\cos x| = \cos u$$

Thus:

$$\cos u = u$$

The function  $\cos u - u$  is strictly decreasing on  $[0, \frac{\pi}{2})$  because:

$$\frac{d}{du}(\cos u - u) = -\sin u - 1 < 0$$

Also:

$$\cos 0 - 0 = 1 > 0$$

$$\cos \frac{\pi}{2} - \frac{\pi}{2} < 0$$

So there is exactly one solution in this interval.

For  $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ :

$$\tan^{-1}(\tan x) = x$$

The equation becomes:

$$|\cos x| = x$$

Here  $\cos x > 0$ , so:

$$\cos x = x$$

Since the right side must be non-negative,  $x \geq 0$ . On  $[0, \frac{\pi}{2})$ ,  $\cos x - x$  is strictly decreasing, and it changes sign between 0 and 1. Hence there is exactly one solution.

For  $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ :

$$\tan^{-1}(\tan x) = x - \pi$$

The equation becomes:

$$|\cos x| = x - \pi$$

For the right side to be non-negative:

$$x \geq \pi$$

So consider  $x \in [\pi, \frac{3\pi}{2})$ . Put:

$$u = x - \pi$$

Then:

$$u \in [0, \frac{\pi}{2})$$

Also,

$$|\cos x| = |\cos(\pi + u)|$$

$$|\cos x| = |-\cos u|$$

$$|\cos x| = \cos u$$

Thus:

$$\cos u = u$$

As shown earlier, this equation has exactly one solution.

Therefore, total number of real solutions is:

$$1 + 1 + 1 = 3$$

Hence, the answer is 3.

**11 Answer: C**

Ref 14785

**Hard** ~3 min Principal values and inverse trigonometric equations

Let  $x = \frac{6y}{9-y^2}$ . Since  $0 < |y| < 3$ , we have  $9 - y^2 > 0$ . Therefore the sign of  $x$  is the same as the sign of  $y$ .

Case 1:  $0 < y < 3$ .

Here  $x > 0$ , so  $\frac{9-y^2}{6y} = \frac{1}{x} > 0$ . Since  $\cot^{-1}$  has range  $(0, \pi)$ , for positive argument we have

$$\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1}(x)$$

Thus the equation becomes

$$\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{2\pi}{3}$$

$$2 \tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{2\pi}{3}$$

$$\tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{\pi}{3}$$

Taking tangent on both sides,

$$\frac{6y}{9-y^2} = \sqrt{3}$$

$$6y = \sqrt{3}(9 - y^2)$$

$$\sqrt{3}y^2 + 6y - 9\sqrt{3} = 0$$

Solving,

$$y = \frac{-6 \pm \sqrt{36 + 108}}{2\sqrt{3}}$$

$$y = \frac{-6 \pm 12}{2\sqrt{3}}$$

The root satisfying  $0 < y < 3$  is

$$y = \frac{6}{2\sqrt{3}}$$

$$y = \sqrt{3}$$

Case 2:  $-3 < y < 0$ .

Here  $x < 0$ , so  $\frac{1}{x} < 0$ . For negative argument, because  $\cot^{-1}$  has range  $(0, \pi)$ , we get

$$\cot^{-1}\left(\frac{1}{x}\right) = \pi + \tan^{-1}(x)$$

Therefore,

$$\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \pi + \tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{2\pi}{3}$$

$$2 \tan^{-1}\left(\frac{6y}{9-y^2}\right) = -\frac{\pi}{3}$$

$$\tan^{-1}\left(\frac{6y}{9-y^2}\right) = -\frac{\pi}{6}$$

Taking tangent on both sides,

$$\frac{6y}{9-y^2} = -\frac{1}{\sqrt{3}}$$

$$6\sqrt{3}y = -9 + y^2$$

$$y^2 - 6\sqrt{3}y - 9 = 0$$

Solving,

$$y = \frac{6\sqrt{3} \pm \sqrt{108 + 36}}{2}$$

$$y = \frac{6\sqrt{3} \pm 12}{2}$$

The root satisfying  $-3 < y < 0$  is

$$y = 3\sqrt{3} - 6$$

Therefore, the sum of all solutions is

$$\begin{aligned} & \sqrt{3} + 3\sqrt{3} - 6 \\ &= 4\sqrt{3} - 6 \end{aligned}$$

Hence, the correct answer is option C.

#### COMMON TRAPS

- **Treating cot inverse reciprocal identity as branch-free.** Given  $\cot^{-1}(y) \in (0, \pi)$ ,  $\cot^{-1}(1/x) = \tan^{-1}(x)$  only for  $x > 0$ ; for  $x < 0$ , it is  $\pi + \tan^{-1}(x)$ . *Avoid it:* Before applying inverse trig identities, write the interval in which each inverse function must lie and test with a negative numerical value, e.g.  $x = -1$ .
- **Ignoring the negative-y case in the domain.**  $0 < |y| < 3$  means  $-3 < y < 0$  or  $0 < y < 3$ , excluding  $y = 0$ . *Avoid it:* Split absolute-value domains explicitly before solving:  $(-3, 0) \cup (0, 3)$ .

**REVISE** Principal ranges of  $\tan^{-1} x$  and  $\cot^{-1} x$ , especially for negative arguments. · Before solving inverse trig equations, split domains and determine signs of all substituted quantities.

#### 12 Answer: $3\pi/4$

Ref 15107

Hard ~3 min Principal values of inverse trigonometric functions

**QUICK APPROACH** Start by expressing the given inverse trigonometric terms in terms of a single angle  $\theta$  using trigonometric identities and the principal value ranges.

Let

$$\theta = \tan^{-1} \left( \frac{\pi}{\sqrt{2}} \right)$$

Since  $\frac{\pi}{\sqrt{2}} > 0$ , we have

$$0 < \theta < \frac{\pi}{2}$$

Now,

$$\cos \theta = \frac{1}{\sqrt{1 + \tan^2 \theta}}$$

$$\cos \theta = \frac{1}{\sqrt{1 + \frac{\pi^2}{2}}}$$

$$\cos \theta = \sqrt{\frac{2}{2 + \pi^2}}$$

Therefore,

$$\cos^{-1} \sqrt{\frac{2}{2 + \pi^2}} = \theta$$

Also,

$$\sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

$$\sin 2\theta = \frac{2 \cdot \frac{\pi}{\sqrt{2}}}{1 + \frac{\pi^2}{2}}$$

$$\sin 2\theta = \frac{2\sqrt{2}\pi}{2 + \pi^2}$$

Now, since  $\tan \theta = \frac{\pi}{\sqrt{2}} > 1$ , we get

$$\theta > \frac{\pi}{4}$$

So,

$$\frac{\pi}{2} < 2\theta < \pi$$

The principal range of  $\sin^{-1} x$  is  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ . Hence,

$$\sin^{-1}(\sin 2\theta) = \pi - 2\theta$$

So,

$$\sin^{-1} \left( \frac{2\sqrt{2}\pi}{2 + \pi^2} \right) = \pi - 2\theta$$

Also,

$$\tan^{-1} \left( \frac{\sqrt{2}}{\pi} \right) = \frac{\pi}{2} - \theta$$

Now substitute these in the given expression.

$$E = \frac{3}{2}\theta + \frac{1}{4}(\pi - 2\theta) + \left(\frac{\pi}{2} - \theta\right)$$

$$E = \frac{3}{2}\theta + \frac{\pi}{4} - \frac{\theta}{2} + \frac{\pi}{2} - \theta$$

The coefficient of  $\theta$  becomes zero.

$$E = \frac{\pi}{4} + \frac{\pi}{2}$$

$$E = \frac{3\pi}{4}$$

Therefore, the value of the expression is  $\frac{3\pi}{4}$ .

13 Answer: 0.5

Ref 15108

Medium ~2 min Limits of composite functions and logarithmic limits

**QUICK APPROACH** Start by substituting the limit variable and use the properties of logarithms and limits to simplify the expression inside the composite function before applying continuity of the sine function.

Let

$$t = \sqrt{x} - \sqrt{\alpha}$$

As  $x \rightarrow \alpha^+$ , we have

$$t \rightarrow 0^+$$

Also,

$$\sqrt{x} = \sqrt{\alpha} + t$$

Now simplify the denominator of  $g(x)$ :

$$e^{\sqrt{x}} - e^{\sqrt{\alpha}} = e^{\sqrt{\alpha}+t} - e^{\sqrt{\alpha}}$$

$$e^{\sqrt{x}} - e^{\sqrt{\alpha}} = e^{\sqrt{\alpha}}(e^t - 1)$$

Taking logarithm,

$$\log_e (e^{\sqrt{x}} - e^{\sqrt{\alpha}}) = \log_e (e^{\sqrt{\alpha}}(e^t - 1))$$

$$\log_e (e^{\sqrt{x}} - e^{\sqrt{\alpha}}) = \sqrt{\alpha} + \log_e (e^t - 1)$$

Also,

$$\sqrt{x} - \sqrt{\alpha} = t$$

So,

$$g(x) = \frac{2 \log_e t}{\sqrt{\alpha} + \log_e (e^t - 1)}$$

Now use

$$e^t - 1 = t \cdot \frac{e^t - 1}{t}$$

Therefore,

$$\log_e (e^t - 1) = \log_e t + \log_e \left( \frac{e^t - 1}{t} \right)$$

As  $t \rightarrow 0^+$ ,

$$\frac{e^t - 1}{t} \rightarrow 1$$

Hence,

$$\log_e \left( \frac{e^t - 1}{t} \right) \rightarrow 0$$

So,

$$g(x) = \frac{2 \log_e t}{\sqrt{\alpha} + \log_e t + \log_e \left( \frac{e^t - 1}{t} \right)}$$

Divide numerator and denominator by  $\log_e t$ :

$$g(x) = \frac{2}{\frac{\sqrt{\alpha}}{\log_e t} + 1 + \frac{\log_e \left( \frac{e^t - 1}{t} \right)}{\log_e t}}$$

As  $t \rightarrow 0^+$ ,

$$\log_e t \rightarrow -\infty$$

Thus,

$$\frac{\sqrt{\alpha}}{\log_e t} \rightarrow 0$$

and

$$\frac{\log_e \left( \frac{e^t - 1}{t} \right)}{\log_e t} \rightarrow 0$$

Therefore,

$$\lim_{x \rightarrow \alpha^+} g(x) = 2$$

Since  $f$  is continuous,

$$\lim_{x \rightarrow \alpha^+} f(g(x)) = f(2)$$

Now,

$$f(2) = \sin \left( \frac{2\pi}{12} \right)$$

$$f(2) = \sin \left( \frac{\pi}{6} \right)$$

$$f(2) = \frac{1}{2}$$

Hence, the required value is

0.5

## 14 Answer: C

Ref 15058

Hard ~3 min One-one and onto functions using monotonicity

We need check whether  $f$  is injective and surjective.

For  $x \geq 0$ , we have  $|x| = x$ , so

$$f(x) = x(x - \sin x)$$

For  $x < 0$ , we have  $|x| = -x$ , so

$$f(x) = -x(x - \sin x)$$

First observe that  $f$  is odd. For  $x > 0$ ,

$$f(-x) = |-x|(-x - \sin(-x))$$

$$f(-x) = x(-x + \sin x)$$

$$f(-x) = -x(x - \sin x)$$

$$f(-x) = -f(x)$$

Now we prove strict monotonicity.

For  $x > 0$ ,

$$f(x) = x^2 - x \sin x$$

$$f'(x) = 2x - \sin x - x \cos x$$

$$f'(x) = x(1 - \cos x) + (x - \sin x)$$

For  $x > 0$ ,  $1 - \cos x \geq 0$  and  $x - \sin x > 0$ . Hence

$$f'(x) > 0$$

For  $x < 0$ ,

$$f(x) = -x^2 + x \sin x$$

$$f'(x) = -2x + \sin x + x \cos x$$

$$f'(x) = x(\cos x - 1) - (x - \sin x)$$

For  $x < 0$ , we have  $\cos x - 1 \leq 0$ , so  $x(\cos x - 1) \geq 0$ . Also, for  $x < 0$ ,  $\sin x > x$ , so  $x - \sin x < 0$ , and therefore

$$-(x - \sin x) > 0$$

Thus

$$f'(x) > 0$$

So  $f$  is strictly increasing on  $(-\infty, \infty)$ . Therefore,  $f$  is one-one.

Now check onto. Since  $f$  is continuous and

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

because the dominant term is  $x^2$ , and

$$\lim_{x \rightarrow -\infty} f(x) = -\infty$$

because  $f$  is odd and grows without bound in magnitude.

Hence the range of  $f$  is all of  $\mathbb{R}$ . Therefore,  $f$  is onto.

So  $f$  is both one-one and onto.

## 15 Answer: A

Ref 15059

Hard ~4 min Area bounded by curves involving exponential functions

We first write  $f(x)$  piecewise.

For  $x \leq 1$ ,

$$|x - 1| = 1 - x$$

So

$$e^{-|x-1|} = e^{-(1-x)}$$

$$e^{-|x-1|} = e^{x-1}$$

Hence

$$f(x) = e^{x-1} - e^{x-1}$$

$$f(x) = 0$$

For  $x > 1$ ,

$$|x - 1| = x - 1$$

So

$$e^{-|x-1|} = e^{1-x}$$

Hence

$$f(x) = e^{x-1} - e^{1-x}$$

Therefore,

$$f(x) = \begin{cases} 0, & x \leq 1 \\ e^{x-1} - e^{1-x}, & x > 1 \end{cases}$$

The curve  $g(x)$  is

$$g(x) = \frac{1}{2}(e^{x-1} + e^{1-x})$$

The bounded region starts from  $x = 0$ . From  $x = 0$  to  $x = 1$ , the lower curve is  $f(x) = 0$  and upper curve is  $g(x)$ .

For  $x > 1$ , the region is between  $g(x)$  and  $f(x)$  until they intersect.

Find the intersection for  $x > 1$ :

$$f(x) = g(x)$$

$$e^{x-1} - e^{1-x} = \frac{1}{2}(e^{x-1} + e^{1-x})$$

Multiply by 2:

$$2e^{x-1} - 2e^{1-x} = e^{x-1} + e^{1-x}$$

$$e^{x-1} = 3e^{1-x}$$

$$e^{2x-2} = 3$$

$$2x - 2 = \ln 3$$

$$x = 1 + \frac{1}{2} \ln 3$$

Let

$$\alpha = 1 + \frac{1}{2} \ln 3$$

At  $x = \alpha$ ,

$$e^{\alpha-1} = \sqrt{3}$$

$$e^{1-\alpha} = \frac{1}{\sqrt{3}}$$

Required area is

$$A = \int_0^1 g(x) dx + \int_1^\alpha \{g(x) - f(x)\} dx$$

First integral:

$$\int_0^1 g(x) dx = \frac{1}{2} \int_0^1 (e^{x-1} + e^{1-x}) dx$$

$$= \frac{1}{2} [e^{x-1} - e^{1-x}]_0^1$$

$$= \frac{1}{2} [(1-1) - (e^{-1} - e)]$$

$$= \frac{1}{2}(e - e^{-1})$$

Now the second integral:

$$g(x) - f(x) = \frac{1}{2}(e^{x-1} + e^{1-x}) - (e^{x-1} - e^{1-x})$$

$$g(x) - f(x) = -\frac{1}{2}e^{x-1} + \frac{3}{2}e^{1-x}$$

So

$$\begin{aligned} \int_1^\alpha \{g(x) - f(x)\} dx &= \int_1^\alpha \left(-\frac{1}{2}e^{x-1} + \frac{3}{2}e^{1-x}\right) dx \\ &= \left[-\frac{1}{2}e^{x-1} - \frac{3}{2}e^{1-x}\right]_1^\alpha \end{aligned}$$

At  $x = \alpha$ ,

$$-\frac{1}{2}e^{\alpha-1} - \frac{3}{2}e^{1-\alpha} = -\frac{\sqrt{3}}{2} - \frac{3}{2\sqrt{3}}$$

$$= -\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}$$

$$= -\sqrt{3}$$

At  $x = 1$ ,

$$-\frac{1}{2}e^0 - \frac{3}{2}e^0 = -2$$

Therefore,

$$\int_1^\alpha \{g(x) - f(x)\} dx = -\sqrt{3} - (-2)$$

$$= 2 - \sqrt{3}$$

Hence total area is

$$A = (2 - \sqrt{3}) + \frac{1}{2}(e - e^{-1})$$

Therefore, the correct option is A.

**16 Answer: 19.00**

Ref 15105

Medium ~2 min Functional equations and symmetric summation

**QUICK APPROACH** Start by exploring the symmetry property of the function  $f(x)$  and find the relation between  $f(x)$  and  $f(1-x)$ .

We need to evaluate the finite sum using the symmetry of the function.

Given:

$$f(x) = \frac{4^x}{4^x + 2}$$

Now find  $f(1-x)$ .

$$f(1-x) = \frac{4^{1-x}}{4^{1-x} + 2}$$

Let  $4^x = a$ .

Then:

$$4^{1-x} = \frac{4}{a}$$

So:

$$f(1-x) = \frac{\frac{4}{a}}{\frac{4}{a} + 2}$$

$$f(1-x) = \frac{4}{4 + 2a}$$

$$f(1-x) = \frac{2}{a + 2}$$

Also:

$$f(x) = \frac{a}{a + 2}$$

Therefore:

$$f(x) + f(1-x) = \frac{a}{a+2} + \frac{2}{a+2}$$

$$f(x) + f(1-x) = 1$$

Now let:

$$S = \sum_{r=1}^{39} f\left(\frac{r}{40}\right)$$

Pair the terms  $\frac{r}{40}$  and  $\frac{40-r}{40}$ , because they add to 1.

Using the identity:

$$f\left(\frac{r}{40}\right) + f\left(\frac{40-r}{40}\right) = 1$$

Hence:

$$2S = \sum_{r=1}^{39} \left[ f\left(\frac{r}{40}\right) + f\left(\frac{40-r}{40}\right) \right]$$

$$2S = 39$$

$$S = \frac{39}{2}$$

Now calculate  $f\left(\frac{1}{2}\right)$ .

$$4^{1/2} = 2$$

$$f\left(\frac{1}{2}\right) = \frac{2}{2+2}$$

$$f\left(\frac{1}{2}\right) = \frac{1}{2}$$

Required value:

$$S - f\left(\frac{1}{2}\right) = \frac{39}{2} - \frac{1}{2}$$

$$S - f\left(\frac{1}{2}\right) = 19$$

So the answer is 19.00.

**17 Answer: C, D**

Ref 14594

Medium ~2 min Limits involving modulus and one-sided limits

For each option, we check the corresponding limit at  $h = 0$ .

Since all the given functions have  $f(0) = 0$ , the expressions simplify directly.

Option A:  $f(x) = x|x|$  and PROPERTY 2.

$$f(0) = 0$$

$$\frac{f(h) - f(0)}{h^2} = \frac{h|h|}{h^2}$$

For  $h > 0$ ,

$$\frac{h|h|}{h^2} = 1$$

For  $h < 0$ ,

$$\frac{h|h|}{h^2} = -1$$

The left-hand and right-hand limits are different. Hence the limit does not exist. So A is incorrect.

Option B:  $f(x) = \sin x$  and PROPERTY 2.

$$f(0) = 0$$

$$\frac{f(h) - f(0)}{h^2} = \frac{\sin h}{h^2}$$

Rewrite it as:

$$\frac{\sin h}{h^2} = \frac{\sin h}{h} \cdot \frac{1}{h}$$

As  $h \rightarrow 0$ ,

$$\frac{\sin h}{h} \rightarrow 1$$

but

$$\frac{1}{h}$$

is unbounded and has different behavior from the two sides. Therefore the limit is not finite. So B is incorrect.

Option C:  $f(x) = |x|$  and PROPERTY 1.

$$f(0) = 0$$

$$\frac{f(h) - f(0)}{\sqrt{|h|}} = \frac{|h|}{\sqrt{|h|}}$$

$$\frac{|h|}{\sqrt{|h|}} = \sqrt{|h|}$$

As  $h \rightarrow 0$ ,

$$\sqrt{|h|} \rightarrow 0$$

The limit exists and is finite. So C is correct.

Option D:  $f(x) = x^{2/3}$  and PROPERTY 1. For real  $x$ ,  $x^{2/3}$  is non-negative and equals  $|x|^{2/3}$ .

$$f(0) = 0$$

$$\frac{f(h) - f(0)}{\sqrt{|h|}} = \frac{|h|^{2/3}}{|h|^{1/2}}$$

$$\frac{|h|^{2/3}}{|h|^{1/2}} = |h|^{1/6}$$

As  $h \rightarrow 0$ ,

$$|h|^{1/6} \rightarrow 0$$

The limit exists and is finite. So D is correct.

Therefore, the correct options are C and D.

#### COMMON TRAPS

• **Treating PROPERTY 2 as ordinary differentiability by denominator  $h$  instead of  $h^2$ .** PROPERTY 2 requires the increment to be of order at most  $h^2$ , not just order  $h$ . For  $\sin h \sim h$ , dividing by  $h^2$  gives roughly  $1/h$ , not a finite limit. *Avoid it:* Before using any standard limit, underline the denominator: is it  $h$ ,  $h^2$ , or  $\sqrt{|h|}$ ?

• **Ignoring one-sided signs in  $x|x|$  divided by  $h^2$ .**  $|h| = h$  for  $h > 0$  but  $|h| = -h$  for  $h < 0$ . Thus  $h|h|/h^2 = 1$  from the right and  $-1$  from the left. *Avoid it:* Whenever an expression contains  $|h|$  and  $h \rightarrow 0$ , explicitly test  $h > 0$  and  $h < 0$ .

**REVIEW** For every limit with  $|h|$ , write separate  $h > 0$  and  $h < 0$  cases unless the expression is already purely in  $|h|$ . • Memorize order comparison near zero: if  $f(h) \sim |h|^a$  and denominator is  $|h|^b$ , the ratio behaves like  $|h|^{a-b}$ .

#### 18 Answer: 2

Ref 14438

Hard ~4 min

For  $x \in (-\frac{1}{2}, \frac{1}{2})$ , all the given infinite geometric series are convergent.

First simplify the left-hand side argument.

$$\sum_{i=1}^{\infty} x^{i+1} = x^2 + x^3 + x^4 + \dots$$

$$\sum_{i=1}^{\infty} x^{i+1} = \frac{x^2}{1-x}$$

Also,

$$\sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^i = \frac{\frac{x}{2}}{1-\frac{x}{2}}$$

$$\sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^i = \frac{x}{2-x}$$

Therefore,

$$\sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^i = \frac{x^2}{1-x} - \frac{x^2}{2-x}$$

$$\sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^i = x^2 \left(\frac{1}{1-x} - \frac{1}{2-x}\right)$$

$$\sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left(\frac{x}{2}\right)^i = \frac{x^2}{(1-x)(2-x)}$$

Now simplify the right-hand side argument.

$$\sum_{i=1}^{\infty} \left(-\frac{x}{2}\right)^i = \frac{-\frac{x}{2}}{1+\frac{x}{2}}$$

$$\sum_{i=1}^{\infty} \left(-\frac{x}{2}\right)^i = \frac{-x}{2+x}$$

Also,

$$\sum_{i=1}^{\infty} (-x)^i = \frac{-x}{1+x}$$

Hence,

$$\sum_{i=1}^{\infty} \left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty} (-x)^i = \frac{-x}{2+x} - \left(\frac{-x}{1+x}\right)$$

$$\sum_{i=1}^{\infty} \left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty} (-x)^i = x \left(\frac{1}{1+x} - \frac{1}{2+x}\right)$$

$$\sum_{i=1}^{\infty} \left(-\frac{x}{2}\right)^i - \sum_{i=1}^{\infty} (-x)^i = \frac{x}{(1+x)(2+x)}$$

So the equation becomes

$$\sin^{-1} \left( \frac{x^2}{(1-x)(2-x)} \right) = \frac{\pi}{2} - \cos^{-1} \left( \frac{x}{(1+x)(2+x)} \right)$$

Using the identity

$$\frac{\pi}{2} - \cos^{-1} y = \sin^{-1} y$$

we get

$$\sin^{-1} \left( \frac{x^2}{(1-x)(2-x)} \right) = \sin^{-1} \left( \frac{x}{(1+x)(2+x)} \right)$$

Since  $\sin^{-1} x$  is one-one on its principal domain,

$$\frac{x^2}{(1-x)(2-x)} = \frac{x}{(1+x)(2+x)}$$

This gives

$$x \left( \frac{x}{(1-x)(2-x)} - \frac{1}{(1+x)(2+x)} \right) = 0$$

Therefore one solution is

$$x = 0$$

For  $x \neq 0$ ,

$$x(1+x)(2+x) = (1-x)(2-x)$$

Expanding both sides,

$$x(x^2 + 3x + 2) = x^2 - 3x + 2$$

$$x^3 + 3x^2 + 2x = x^2 - 3x + 2$$

$$x^3 + 2x^2 + 5x - 2 = 0$$

Let

$$f(x) = x^3 + 2x^2 + 5x - 2$$

Then

$$f'(x) = 3x^2 + 4x + 5$$

The discriminant of  $3x^2 + 4x + 5$  is

$$4^2 - 4(3)(5) = -44$$

Since the leading coefficient is positive and the discriminant is negative,

$$f'(x) > 0$$

So  $f(x)$  is strictly increasing for all real  $x$ .

Now,

$$f(0) = -2$$

and

$$f\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{1}{2} + \frac{5}{2} - 2$$

$$f\left(\frac{1}{2}\right) = \frac{9}{8} > 0$$

Thus, by the intermediate value theorem, there is exactly one root of  $f(x) = 0$  in  $(0, \frac{1}{2})$ .

Because  $f(x)$  is strictly increasing, this root is unique. Along with  $x = 0$ , the total number of solutions in  $(-\frac{1}{2}, \frac{1}{2})$  is

2

Therefore, the required answer is 2.

**19 Answer: A**

Hard ~3 min

Ref 14494

First find  $E_1$ .

$$\frac{x}{x-1} > 0$$

The critical points are 0 and 1. The expression is positive for  $x < 0$  and for  $x > 1$ .

$$E_1 = (-\infty, 0) \cup (1, \infty)$$

Now put

$$t = \frac{x}{x-1}$$

For  $x \in (-\infty, 0)$ , we get  $0 < t < 1$ . For  $x \in (1, \infty)$ , we get  $t > 1$ .

$$t \in (0, 1) \cup (1, \infty)$$

Since

$$f(x) = \log_e t$$

we get

$$\text{Range}(f) = (-\infty, 0) \cup (0, \infty)$$

So P matches 4.

For  $g(x)$  to be real, the argument of  $\sin^{-1}$  must lie in  $[-1, 1]$ .

$$-1 \leq \log_e \left( \frac{x}{x-1} \right) \leq 1$$

Using  $t = \frac{x}{x-1}$ ,

$$e^{-1} \leq t \leq e$$

Also,  $t \neq 1$  because  $\frac{x}{x-1} = 1$  has no finite solution.

So

$$t \in \left[ \frac{1}{e}, 1 \right) \cup (1, e]$$

From

$$t = \frac{x}{x-1}$$

we get

$$x = \frac{t}{t-1}$$

For  $t \in \left[ \frac{1}{e}, 1 \right)$ ,

$$x \in \left( -\infty, -\frac{1}{e-1} \right]$$

Since

$$-\frac{1}{e-1} = \frac{1}{1-e}$$

this becomes

$$x \in \left( -\infty, \frac{1}{1-e} \right]$$

For  $t \in (1, e]$ ,

$$x \in \left[ \frac{e}{e-1}, \infty \right)$$

Therefore

$$E_2 = (-\infty, \frac{1}{1-e}] \cup [\frac{e}{e-1}, \infty)$$

Thus the domain of  $g$  is item 1, so S matches 1.

The domain of  $f$  is  $E_1 = (-\infty, 0) \cup (1, \infty)$ . Item 1 is fully contained in  $E_1$ , hence R matches 1.

Now find what the range of  $g$  contains. Since

$$\log_e \left( \frac{x}{x-1} \right) \in [-1, 0) \cup (0, 1]$$

and  $\sin^{-1} u$  is increasing on  $[-1, 1]$ ,

$$\text{Range}(g) = \left[ -\frac{\pi}{2}, 0 \right) \cup \left( 0, \frac{\pi}{2} \right]$$

This range contains the interval  $(0, 1)$ , but it does not contain  $\left[ -\frac{1}{2}, \frac{1}{2} \right]$  because 0 is missing from the range. Hence Q matches 2.

Therefore the correct matching is:

$$P \rightarrow 4, Q \rightarrow 2, R \rightarrow 1, S \rightarrow 1$$

So the correct option is A.

**20 Answer: A, B, C**

Ref 13960

**Hard** ~3 min Odd-even functions, one-one and onto functions

Let  $g(x) = \log(\sec x + \tan x)$ . First check oddness. For  $x \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right)$ ,  $\sec x + \tan x > 0$ , so  $g(x)$  is defined.

Now,

$$g(-x) = \log(\sec(-x) + \tan(-x))$$

$$g(-x) = \log(\sec x - \tan x)$$

Using

$$(\sec x + \tan x)(\sec x - \tan x) = \sec^2 x - \tan^2 x$$

$$\sec^2 x - \tan^2 x = 1$$

So,

$$\sec x - \tan x = \frac{1}{\sec x + \tan x}$$

Hence,

$$g(-x) = \log\left(\frac{1}{\sec x + \tan x}\right)$$

$$g(-x) = -\log(\sec x + \tan x)$$

$$g(-x) = -g(x)$$

Therefore  $g$  is odd. Since  $f(x) = (g(x))^3$ , we get

$$f(-x) = (g(-x))^3$$

$$f(-x) = (-g(x))^3$$

$$f(-x) = -(g(x))^3$$

$$f(-x) = -f(x)$$

So  $f$  is odd. Option A is correct, and option D is incorrect.

Now check one-one nature. Differentiate  $g(x)$ :

$$g'(x) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}$$

$$g'(x) = \frac{\sec x(\tan x + \sec x)}{\sec x + \tan x}$$

$$g'(x) = \sec x$$

$$\text{On } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

$$\sec x > 0$$

So  $g$  is strictly increasing. The cube function is also strictly increasing. Hence  $f(x) = (g(x))^3$  is strictly increasing, so it is one-one. Option B is correct.

Now check onto. As  $x \rightarrow -\frac{\pi}{2}^+$ ,

$$\sec x + \tan x \rightarrow 0^+$$

So,

$$g(x) = \log(\sec x + \tan x) \rightarrow -\infty$$

Therefore,

$$f(x) = (g(x))^3 \rightarrow -\infty$$

$$\text{As } x \rightarrow \frac{\pi}{2}^-,$$

$$\sec x + \tan x \rightarrow \infty$$

So,

$$g(x) \rightarrow \infty$$

Therefore,

$$f(x) \rightarrow \infty$$

Since  $f$  is continuous and strictly increasing from  $-\infty$  to  $\infty$ , its range is  $\mathbb{R}$ . Hence  $f$  is onto. Option C is correct.

Thus the correct options are A, B and C.

**21 Answer: A, B, C**

Ref 13984

Hard

~3 min

One-one, onto and odd functions

Check symmetry, monotonicity and the range of  $\log(\sec x + \tan x)$ .

**Step 1.**  $\sec(-x) + \tan(-x) = \sec x - \tan x = \frac{1}{\sec x + \tan x}$ , so  $\log(\sec(-x) + \tan(-x)) = -\log(\sec x + \tan x)$  and  $f(-x) = -f(x)$ . So  $f$  is odd and, being nonzero, not even.

**Step 2.**  $\frac{d}{dx}(\sec x + \tan x) = \sec x(\sec x + \tan x) > 0$  on  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ , so the log and its cube are strictly increasing. So  $f$  is one-one.

**Step 3.**  $\sec x + \tan x = \frac{1+\sin x}{\cos x}$  tends to  $0^+$  as  $x \rightarrow -\frac{\pi}{2}^+$  and to  $\infty$  as  $x \rightarrow \frac{\pi}{2}^-$ . So the log takes every real value, and so does its cube:  $f$  is onto  $\mathbb{R}$ .

Answer: (A), (B), (C) odd, one-one and onto

#### COMMON TRAPS

- **Expecting a bounded domain to give a bounded range.** On  $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$  the value  $\sec x + \tan x = \frac{1+\sin x}{\cos x}$  runs from  $0^+$  to  $\infty$ , so its log takes every real value and  $f$  is onto  $\mathbb{R}$ . *Avoid it:* Find the limits of the function at both open ends of the interval before judging the range.
- **Missing the reciprocal identity in the parity check.** Since  $(\sec x + \tan x)(\sec x - \tan x) = 1$ ,  $\log(\sec x - \tan x) = -\log(\sec x + \tan x)$ , and the cube of an odd function is odd, so  $f(-x) = -f(x)$ . *Avoid it:* Rewrite  $\sec x - \tan x$  as  $\frac{1}{\sec x + \tan x}$  before comparing  $f(-x)$  with  $f(x)$ .

**REVIEW** The identity  $\sec^2 x - \tan^2 x = 1$  and its reciprocal form. Testing one-one and onto through derivative sign and limits.

**22 Answer: A**

Ref 14020

Hard

~1 min

Function evaluation

At  $a = \frac{1}{2}$  the integrand is  $\frac{1}{\sqrt{t(1-t)}}$ , which a sine substitution integrates.

$$\text{Step 1. } g\left(\frac{1}{2}\right) = \int_0^1 \frac{dt}{\sqrt{t(1-t)}}$$

**Step 2.** Put  $t = \sin^2 \theta$ ,  $dt = 2 \sin \theta \cos \theta d\theta$ ,  $\sqrt{t(1-t)} = \sin \theta \cos \theta$ . The integral becomes  $\int_0^{\pi/2} 2 d\theta = \pi$ .

Answer: (A)  $\pi$

## COMMON TRAPS

- **Confusing  $g(1/2)$  with a standard inverse trigonometric value.** Function notation uses arbitrary names.  $g$  has no default meaning unless defined in the passage/question. *Avoid it:* Circle the actual function symbol and locate its definition; do not replace  $g$  by a familiar function just because the options look trigonometric.

**REVISE** Practice identifying whether a function is actually defined before evaluation. · For passage-based questions, always verify the question-number range of the passage.

23 Answer: A, B

Ref 13910

Hard ~3 min Absolute value functions and local extrema

Let

$$a = 2|x|$$

and

$$b = |x + 2|$$

The given function is

$$f(x) = a + b - |b - a|$$

Using the identity

$$a + b - |a - b| = 2 \min(a, b)$$

we get

$$f(x) = 2 \min(2|x|, |x + 2|)$$

Multiplication by the positive constant 2 does not change the points of local maxima or local minima. So we study

$$h(x) = \min(2|x|, |x + 2|)$$

The two expressions are equal when

$$2|x| = |x + 2|$$

Squaring both sides,

$$4x^2 = (x + 2)^2$$

$$4x^2 = x^2 + 4x + 4$$

$$3x^2 - 4x - 4 = 0$$

$$(3x + 2)(x - 2) = 0$$

So the intersection points are

$$x = -\frac{2}{3}$$

and

$$x = 2$$

Also, because of modulus functions, possible corner points are

$$x = 0$$

and

$$x = -2$$

Now examine intervals.

For  $x < -2$ ,

$$h(x) = |x + 2| = -x - 2$$

This decreases up to  $x = -2$ .For  $-2 < x < -\frac{2}{3}$ ,

$$h(x) = |x + 2| = x + 2$$

This increases after  $x = -2$ . Hence  $x = -2$  is a local minimum.

For  $-\frac{2}{3} < x < 0$ ,

$$h(x) = 2|x| = -2x$$

This decreases after  $x = -\frac{2}{3}$ . Since the function was increasing before  $x = -\frac{2}{3}$  and decreasing after it,  $x = -\frac{2}{3}$  is a local maximum.

At  $x = 0$ , there is also a local minimum, but 0 is not among the given options.

At  $x = 2$ , the function is increasing on both sides, so it is not a local extremum.

Therefore, among the given options, the required values are

$$x = -2$$

and

$$x = -\frac{2}{3}$$

Correct answer: A,B.

## COMMON TRAPS

- **Treating the outer modulus as removable: using  $a+b-(b-a)=2a$  everywhere.**  $|b-a|$  equals  $b-a$  only when  $b \geq a$ , and equals  $a-b$  when  $a > b$ . Here the sign changes at  $x = -2/3$  and  $x = 2$ . *Avoid it:* Before removing an absolute value, solve where its inside is zero and split intervals.
- **Using max instead of min in the identity.** The minus sign before  $|a-b|$  selects the smaller quantity; the plus sign selects the larger quantity. *Avoid it:* Test with numbers  $a=5, b=2$ :  $5+2-|3|=4=2\min$ , not 10.

**REVISE** Piecewise analysis of modulus functions using breakpoint tables. · Practice identifying local extrema of min/max of two V-shaped functions.

## 24 Answer: B

Ref 13732

Medium ~1.5 min One-one and onto functions

We need to check whether the function is injective and surjective from  $[0, 3]$  to  $[1, 29]$ .

First find the derivative to study monotonicity.

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

$$f'(x) = 6x^2 - 30x + 36$$

$$f'(x) = 6(x^2 - 5x + 6)$$

$$f'(x) = 6(x-2)(x-3)$$

Now check signs on  $[0, 3]$ .

For  $0 < x < 2$ , both  $x-2$  and  $x-3$  are negative, so their product is positive.

Therefore,  $f'(x) > 0$  on  $(0, 2)$ , so  $f$  is increasing on  $[0, 2]$ .

For  $2 < x < 3$ ,  $x-2$  is positive and  $x-3$  is negative, so their product is negative.

Therefore,  $f'(x) < 0$  on  $(2, 3)$ , so  $f$  is decreasing on  $[2, 3]$ .

Now compute the endpoint and critical point values.

$$f(0) = 2(0)^3 - 15(0)^2 + 36(0) + 1$$

$$f(0) = 1$$

$$f(2) = 2(2)^3 - 15(2)^2 + 36(2) + 1$$

$$f(2) = 16 - 60 + 72 + 1$$

$$f(2) = 29$$

$$f(3) = 2(3)^3 - 15(3)^2 + 36(3) + 1$$

$$f(3) = 54 - 135 + 108 + 1$$

$$f(3) = 28$$

Since the function is continuous, increases from 1 to 29 on  $[0, 2]$ , and then decreases from 29 to 28 on  $[2, 3]$ , its range is

exactly  $[1, 29]$ .

So the function is onto.

But it is not one-one because on the interval  $(2, 3)$  the function repeats values already taken on the increasing part  $[0, 2]$ . For example, every value between 28 and 29 is taken once before  $x = 2$  and once after  $x = 2$ .

Therefore, the function is onto but not one-one.

Correct option: B

## COMMON TRAPS

- **Treating the codomain as the domain.** In  $f : A \rightarrow B$ , inputs are only from  $A$ ; onto is checked by comparing the actual range over  $A$  with  $B$ . *Avoid it:* Before differentiating/evaluating, write: domain =  $[0, 3]$ , codomain =  $[1, 29]$ .
- **Forgetting that onto is with respect to the given codomain.** Surjectivity means range = declared codomain, not range =  $\mathbb{R}$  unless the codomain is  $\mathbb{R}$ . *Avoid it:* Circle the codomain in the question and ask whether every number in that exact interval is hit.

**REVISE** Domain-codomain-range distinction for functions written as  $f : A \rightarrow B$ . · Practice closed-interval range problems using derivative sign charts and endpoint checks.

## 25 Answer: B

Ref 13798

Medium ~2 min Definite integrals using symmetry

Let the required integral be  $I$ .

$$I = \int_{-\pi/2}^{\pi/2} \left( x^2 + \ln \frac{\pi+x}{\pi-x} \right) \cos x \, dx$$

Split it into two parts.

$$I = \int_{-\pi/2}^{\pi/2} x^2 \cos x \, dx + \int_{-\pi/2}^{\pi/2} \ln \frac{\pi+x}{\pi-x} \cos x \, dx$$

Now consider

$$f(x) = \ln \frac{\pi+x}{\pi-x}$$

Then

$$f(-x) = \ln \frac{\pi-x}{\pi+x}$$

$$f(-x) = -\ln \frac{\pi+x}{\pi-x}$$

$$f(-x) = -f(x)$$

So  $f(x)$  is an odd function. Also,  $\cos x$  is even. Therefore,  $f(x) \cos x$  is odd.

Hence, over the symmetric interval  $[-\pi/2, \pi/2]$ ,

$$\int_{-\pi/2}^{\pi/2} \ln \frac{\pi+x}{\pi-x} \cos x \, dx = 0$$

Thus,

$$I = \int_{-\pi/2}^{\pi/2} x^2 \cos x \, dx$$

Here,  $x^2$  is even and  $\cos x$  is even, so  $x^2 \cos x$  is even.

$$I = 2 \int_0^{\pi/2} x^2 \cos x \, dx$$

Now integrate by parts.

$$\int x^2 \cos x \, dx = x^2 \sin x - \int 2x \sin x \, dx$$

Also,

$$\int x \sin x \, dx = -x \cos x + \sin x$$

Therefore,

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x$$

Now apply the limits.

$$\int_0^{\pi/2} x^2 \cos x \, dx = [x^2 \sin x + 2x \cos x - 2 \sin x]_0^{\pi/2}$$

At  $x = \frac{\pi}{2}$ ,

$$x^2 \sin x + 2x \cos x - 2 \sin x = \frac{\pi^2}{4} - 2$$

At  $x = 0$ ,

$$x^2 \sin x + 2x \cos x - 2 \sin x = 0$$

So,

$$\int_0^{\pi/2} x^2 \cos x \, dx = \frac{\pi^2}{4} - 2$$

Hence,

$$I = 2 \left( \frac{\pi^2}{4} - 2 \right)$$

$$I = \frac{\pi^2}{2} - 4$$

Therefore, the correct option is B.

#### COMMON TRAPS

- **Treating the logarithmic term as even instead of odd.** Since  $\ln(1/a) = -\ln a$ ,  $f(-x) = \ln \frac{\pi-x}{\pi+x} = -\ln \frac{\pi+x}{\pi-x} = -f(x)$ . *Avoid it:* After substituting  $-x$ , explicitly rewrite the new fraction as the reciprocal of the original fraction before deciding parity.
- **Forgetting that odd times even is odd.** Odd function  $\times$  even function = odd function, and the integral of an odd function over  $[-a, a]$  is 0. *Avoid it:* Mark parity under every factor before multiplying:  $\ln$ -term = O,  $\cos x$  = E, product = O.

**REVIEW** Symmetry rules: odd/even products and integrals over  $[-a, a]$ . Practice recognizing  $\ln((a+x)/(a-x))$  as an odd function.

#### 26 Answer: B, C

Ref 13618

Medium ~2 min Cauchy functional equation and differentiability

Additivity turns the difference quotient at any point into the difference quotient at 0, so differentiability at one point spreads to every point.

**Step 1.** Put  $x = y = 0$ :  $f(0) = 2f(0)$ , so  $f(0) = 0$ . Let  $f'(0) = k$ . This means  $\frac{f(h)}{h} \rightarrow k$  as  $h \rightarrow 0$ .

**Step 2.** For any  $x$ :  $\frac{f(x+h)-f(x)}{h} = \frac{f(x)+f(h)-f(x)}{h} = \frac{f(h)}{h}$ . As  $h \rightarrow 0$  this tends to  $k$ . So  $f'(x) = k$  for every real  $x$ .

**Step 3.** Since  $f'(x) = k$  everywhere,  $f(x) = kx + c$ , and  $f(0) = 0$  gives  $f(x) = kx$ . A differentiable function is continuous, so  $f$  is continuous on all of  $\mathbb{R}$ . Options (B) and (C) are true.

**Step 4.**  $f$  is differentiable at every real number. It is not limited to a finite interval, and there are no exceptional points. Options (A) and (D) are false.

Answer: (B), (C)

#### COMMON TRAPS

- **Thinking differentiable at 0 gives only local differentiability near 0.** For additive  $f$ ,  $f(x+h)-f(x)=f(h)$ , so the derivative at any  $x$  has exactly the same limit as the derivative at 0. *Avoid it:* Before judging options, write  $f'(x) = \lim_{h \rightarrow 0} [f(x+h) - f(x)]/h$  and substitute  $f(x+h)=f(x)+f(h)$ .
- **Forgetting to prove  $f(0)=0$  before using the derivative at 0.** Putting  $x=y=0$  gives  $f(0)=2f(0)$ , hence  $f(0)=0$ . Only then does  $[f(h)-f(0)]/h$  become  $f(h)/h$ . *Avoid it:* Always test functional equations at neutral elements like 0 before differentiating.

**REVIEW** Cauchy functional equation with regularity conditions: continuity/differentiability at one point implies linearity. Practice translating option statements like 'only', 'for all', and 'except finitely many' into precise mathematical set language.

#### 27 Answer: A

Ref 13677

Medium ~1.5 min Composition of functions and trigonometric equations

**QUICK APPROACH** Start by simplifying both sides using the definition of function composition and then use the fact that for a real number  $y$ ,  $y^2 = y$  implies  $y(y-1)=0$ .

We need to simplify both compositions carefully.

Given:

$$f(x) = x^2$$

$$g(x) = \sin x$$

First evaluate the right side:

$$(g \circ g \circ f)(x) = g(g(f(x)))$$

$$f(x) = x^2$$

$$g(f(x)) = \sin(x^2)$$

$$g(g(f(x))) = \sin(\sin(x^2))$$

So,

$$(g \circ g \circ f)(x) = \sin(\sin(x^2))$$

Now evaluate the left side:

$$(f \circ g \circ g \circ f)(x) = f(g(g(f(x))))$$

We already found:

$$g(g(f(x))) = \sin(\sin(x^2))$$

Applying  $f$ , which squares the input:

$$(f \circ g \circ g \circ f)(x) = (\sin(\sin(x^2)))^2$$

The given equation becomes:

$$(\sin(\sin(x^2)))^2 = \sin(\sin(x^2))$$

Let

$$y = \sin(\sin(x^2))$$

Then the equation is:

$$y^2 = y$$

$$y(y-1) = 0$$

So,

$$y = 0 \text{ or } y = 1$$

Now check whether  $y = 1$  is possible.

Since  $\sin(x^2)$  lies between  $-1$  and  $1$ , we have:

$$\sin(x^2) \in [-1, 1]$$

Therefore:

$$\sin(\sin(x^2)) \in [-\sin 1, \sin 1]$$

Since  $\sin 1 < 1$ ,  $y = 1$  is impossible.

Hence only:

$$\sin(\sin(x^2)) = 0$$

For  $\sin u = 0$ , we need:

$$u = k\pi$$

Here,

$$u = \sin(x^2)$$

But  $\sin(x^2) \in [-1, 1]$ , and the only multiple of  $\pi$  in this interval is 0.

So:

$$\sin(x^2) = 0$$

This gives:

$$x^2 = n\pi$$

Since  $x^2 \geq 0$ , we must have:

$$n \in \{0, 1, 2, \dots\}$$

Therefore:

$$x = \pm\sqrt{n\pi}$$

with

$$n \in \{0, 1, 2, \dots\}$$

Hence the correct option is A.

#### COMMON TRAPS

- **Reversing the order of composition.** The question explicitly states " $(f \circ g)(x) = f(g(x))$ ", so  $f \circ g \circ g \circ f$  means first  $f(x) = x^2$ , then  $g$ , then  $g$ , then outer  $f$ . *Avoid it:* Write arrows under the composition:  $x \rightarrow f(x) = x^2 \rightarrow g = \sin(x^2) \rightarrow g = \sin(\sin(x^2)) \rightarrow f = [\sin(\sin(x^2))]^2$ .
- **Dropping the outer square on the left side.** Since  $f(t) = t^2$ , applying the extra outer  $f$  gives  $f(\sin(\sin(x^2))) = [\sin(\sin(x^2))]^2$ . *Avoid it:* Count functions: left has 4 functions  $f, g, g, f$ ; right has 3 functions  $g, g, f$ . The left must differ by one final squaring.

**REVISE** Right-to-left evaluation of composite functions using the definition  $(f \circ g)(x) = f(g(x))$ . Practice equations of the form  $A^2 = A$  without cancelling  $A$ .

28 Answer: A

Hard

~2 min

Invertibility and inverse functions

Ref 13687

We are given

$$f(x) = \frac{b-x}{1-bx}$$

with  $0 < b < 1$  and domain  $(0, 1)$ .

First check the range of  $f$  on  $(0, 1)$ . Since  $1 - bx > 0$  for every  $x \in (0, 1)$ , the function is well-defined.

Differentiate:

$$f'(x) = \frac{-(1-bx) - (b-x)(-b)}{(1-bx)^2}$$

$$f'(x) = \frac{-1 + bx + b^2 - bx}{(1-bx)^2}$$

$$f'(x) = \frac{b^2 - 1}{(1-bx)^2}$$

Since  $0 < b < 1$ , we have  $b^2 - 1 < 0$ . Also  $(1 - bx)^2 > 0$ . Hence

$$f'(x) < 0$$

So  $f$  is strictly decreasing and therefore one-one on  $(0, 1)$ .

Now find its range. At the endpoints of the open interval:

$$\lim_{x \rightarrow 0^+} f(x) = b$$

$$\lim_{x \rightarrow 1^-} f(x) = \frac{b-1}{1-b}$$

$$\lim_{x \rightarrow 1^-} f(x) = -1$$

Since the interval is open and  $f$  is continuous and decreasing, the range is

$$(-1, b)$$

But the given codomain is  $\mathbb{R}$ . For  $f : (0, 1) \rightarrow \mathbb{R}$  to be invertible, it must be bijective onto  $\mathbb{R}$ . It is not onto  $\mathbb{R}$  because its range is only  $(-1, b)$ .

Therefore  $f$  is not invertible as a function from  $(0, 1)$  to  $\mathbb{R}$ .

So option A is correct.

For completeness, if we formally solve for the inverse on its

actual range:

$$y = \frac{b-x}{1-bx}$$

$$y(1-bx) = b-x$$

$$y - byx = b - x$$

$$x(1-by) = b-y$$

$$x = \frac{b-y}{1-by}$$

This has the same algebraic form as  $f$ , but this inverse is defined only on the actual range  $(-1, b)$ , not on the whole interval  $(0, 1)$ . Therefore statements involving  $f^{-1}$  on all of  $(0, 1)$  are not valid. Hence B, C, and D are false.

#### COMMON TRAPS

- **Treating strictly decreasing as automatically invertible onto the stated codomain.** Strictly monotone on the domain gives injectivity only. A function is invertible as a map to its given codomain only if it is bijective onto that codomain. *Avoid it:* After proving one-one, explicitly compute/check the range and compare it with the codomain.
- **Ignoring the codomain  $\mathbb{R}$  in the phrase  $f : (0, 1) \rightarrow \mathbb{R}$ .** The algebraic inverse can exist on the actual range  $(-1, b)$ , but the original function is not onto  $\mathbb{R}$ , so it is not invertible as stated. *Avoid it:* Underline the codomain in the question before solving: here it is  $\mathbb{R}$ , not  $(-1, b)$ .

**REVISE** The distinction between domain, codomain, and range in invertibility questions. • Practise finding ranges of fractional linear functions on open intervals using endpoint limits and monotonicity.

29 Answer: p, q, s, t

Ref 13690

Very hard ~6 min

Vectors, definite integrals, functions and complex numbers

For (A), the given vectors satisfy

$$\vec{a} + \vec{b} = \vec{c}$$

So the triangle is formed by the sides  $\vec{a}$ ,  $\vec{b}$  and  $-\vec{c}$ . First find the angle between  $\vec{a}$  and  $\vec{b}$  when placed tail-to-tail.

$$\vec{a} \cdot \vec{b} = (\hat{j} + \sqrt{3}\hat{k}) \cdot (-\hat{j} + \sqrt{3}\hat{k})$$

$$\vec{a} \cdot \vec{b} = -1 + 3$$

$$\vec{a} \cdot \vec{b} = 2$$

Also,

$$|\vec{a}| = \sqrt{1+3}$$

$$|\vec{a}| = 2$$

$$|\vec{b}| = \sqrt{1+3}$$

$$|\vec{b}| = 2$$

If the angle between the vectors is  $\theta$ , then

$$\cos \theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos \theta = \frac{2}{4}$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

But in the triangle, the internal angle between the corresponding sides is the supplement of this angle because the sides are arranged head-to-tail.

$$\text{Internal angle} = \pi - \frac{\pi}{3}$$

$$\text{Internal angle} = \frac{2\pi}{3}$$

Hence (A) matches (q).

For (B), define

$$F(x) = \int (f(x) - 3x) dx$$

Then the given condition becomes

$$F(b) - F(a) = a^2 - b^2$$

Rearrange:

$$F(b) + b^2 = F(a) + a^2$$

Since this is true for arbitrary  $a$  and  $b$ , the expression  $F(x) + x^2$  must be constant. Differentiate with respect to  $x$ .

$$F'(x) + 2x = 0$$

Since

$$F'(x) = f(x) - 3x$$

we get

$$f(x) - 3x + 2x = 0$$

$$f(x) = x$$

Therefore,

$$f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$$

Hence (B) matches (p).

For (C), let

$$I = \frac{\pi^2}{\ln 3} \int_{-1/6}^{1/6} \sec(\pi x) dx$$

Use the standard result

$$\int \sec u du = \ln |\sec u + \tan u| + C$$

Put

$$u = \pi x$$

Then

$$du = \pi dx$$

So

$$\int \sec(\pi x) dx = \frac{1}{\pi} \ln |\sec(\pi x) + \tan(\pi x)|$$

Now evaluate the definite integral.

$$\sec\left(\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$$

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

So

$$\sec\left(\frac{\pi}{6}\right) + \tan\left(\frac{\pi}{6}\right) = \sqrt{3}$$

Also,

$$\sec\left(-\frac{\pi}{6}\right) = \frac{2}{\sqrt{3}}$$

$$\tan\left(-\frac{\pi}{6}\right) = -\frac{1}{\sqrt{3}}$$

So

$$\sec\left(-\frac{\pi}{6}\right) + \tan\left(-\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

Thus,

$$\int_{-1/6}^{1/6} \sec(\pi x) dx = \frac{1}{\pi} \ln\left(\frac{\sqrt{3}}{1/\sqrt{3}}\right)$$

$$\int_{-1/6}^{1/6} \sec(\pi x) dx = \frac{1}{\pi} \ln 3$$

Therefore,

$$I = \frac{\pi^2}{\ln 3} \cdot \frac{\ln 3}{\pi}$$

$$I = \pi$$

Hence (C) matches (s).

For (D), since  $|z| = 1$ , write

$$z = e^{i\theta}$$

where  $0 < \theta < 2\pi$  because  $z \neq 1$ . Now,

$$1 - z = 1 - e^{i\theta}$$

Using the half-angle form,

$$1 - e^{i\theta} = e^{i\theta/2}(e^{-i\theta/2} - e^{i\theta/2})$$

$$1 - e^{i\theta} = -2ie^{i\theta/2} \sin\left(\frac{\theta}{2}\right)$$

For  $0 < \theta < 2\pi$ ,

$$\sin\left(\frac{\theta}{2}\right) > 0$$

Hence,

$$\text{Arg}(1 - z) = \frac{\theta}{2} - \frac{\pi}{2}$$

Therefore,

$$\text{Arg}\left(\frac{1}{1-z}\right) = -\text{Arg}(1 - z)$$

$$\text{Arg}\left(\frac{1}{1-z}\right) = \frac{\pi}{2} - \frac{\theta}{2}$$

As  $\theta$  varies in  $(0, 2\pi)$ , this argument lies between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$ .

Hence the greatest possible value among the given options is

$$\frac{\pi}{2}$$

Thus (D) matches (t).

Final matching:

$$A \rightarrow q$$

$$B \rightarrow p$$

$$C \rightarrow s$$

$$D \rightarrow t$$

#### COMMON TRAPS

- **Taking the tail-to-tail vector angle as the triangle's internal angle in (A).** Since  $\vec{a} + \vec{b} = \vec{c}$ , the triangle sides are  $\vec{a}$ ,  $\vec{b}$ , and  $-\vec{c}$ ; at the common vertex in the triangle one of  $\vec{a}$ ,  $\vec{b}$  is effectively reversed, so internal angle is  $\pi - \theta$ . *Avoid it:* After finding dot-product angle, sketch head-to-tail arrangement before matching.
- **Wrong sign in  $\vec{a} \cdot \vec{b}$  for (A).** Only like unit vectors contribute; j-term is  $1 \times (-1) = -1$  and k-term is  $\sqrt{3} \times \sqrt{3} = 3$ , so dot product is 2. *Avoid it:* Write component vectors as  $(0, 1, \sqrt{3})$  and  $(0, -1, \sqrt{3})$  before dot product.

**REVIEW** For vector matching, always draw the head-to-tail triangle after dot-product calculations. For integral identities with variable endpoints, practice differentiating after isolating endpoint-dependent terms.

#### 30 Answer: B

Ref 13523

Medium

~1.5 min

Derivative of inverse function

We are given

$$e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4 + 1} dt$$

Multiplying both sides by  $e^x$ , we get

$$f(x) = e^x \left( 2 + \int_0^x \sqrt{t^4 + 1} dt \right)$$

To find  $(f^{-1})'(2)$ , use the inverse derivative formula:

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

where

$$f(x) = y$$

Here,  $y = 2$ . First find  $x$  such that

$$f(x) = 2$$

At  $x = 0$ ,

$$f(0) = e^0 \left( 2 + \int_0^0 \sqrt{t^4 + 1} dt \right)$$

$$f(0) = 1 \cdot (2 + 0)$$

$$f(0) = 2$$

So the required point is  $x = 0$ .

Now differentiate  $f(x)$ .

$$f(x) = e^x \left( 2 + \int_0^x \sqrt{t^4 + 1} dt \right)$$

Using product rule and the Fundamental Theorem of Calculus,

$$f'(x) = e^x \left( 2 + \int_0^x \sqrt{t^4 + 1} dt \right) + e^x \sqrt{x^4 + 1}$$

At  $x = 0$ ,

$$f'(0) = e^0 (2 + 0) + e^0 \sqrt{0^4 + 1}$$

$$f'(0) = 2 + 1$$

$$f'(0) = 3$$

Therefore,

$$(f^{-1})'(2) = \frac{1}{f'(0)}$$

$$(f^{-1})'(2) = \frac{1}{3}$$

Hence, the correct option is B.

#### COMMON TRAPS

- **Treating the input 2 of inverse derivative as  $x=2$ .** For  $(f^{-1})'(y)$ , first find  $x$  such that  $f(x) = y$ , then use  $(f^{-1})'(y) = 1/f'(x)$ . *Avoid it:* Before differentiating, write: target output  $y = 2$ ; solve  $f(x) = 2$ . Here  $f(0) = 2$ , so use  $x = 0$ .
- **Forgetting to multiply by  $e^x$  before finding  $f(x)$ .** From  $e^{-x}f(x) = G(x)$ ,  $f(x) = e^x G(x)$ , not just  $G(x)$ . *Avoid it:* Circle the coefficient multiplying  $f(x)$  and isolate  $f$  before using inverse derivative formula.

**REVISE**  $(f^{-1})'(b) = 1/f'(f^{-1}(b))$  with emphasis on finding the preimage first. • Practice differentiating  $e^x (c + \int_0^x g(t)dt)$  using product rule and FTC.

31 Answer: D

Ref 13524

Medium ~1.5 min Counting unordered pairs of disjoint subsets

We need to count unordered pairs of subsets  $(A, B)$  such that  $A \cap B = \emptyset$ .

First count ordered pairs of disjoint subsets.

For each element of  $S = \{1, 2, 3, 4\}$ , there are exactly three choices:

1. The element goes into  $A$ .
2. The element goes into  $B$ .

3. The element goes into neither  $A$  nor  $B$ .

It cannot go into both because  $A$  and  $B$  must be disjoint.

Since there are 4 elements, the number of ordered pairs is

$$3^4 = 81$$

Now convert ordered pairs into unordered pairs. Usually, each unordered pair corresponds to two ordered pairs:

$(A, B)$  and  $(B, A)$

But there is one special case where swapping does not create a different ordered pair:

$$A = B$$

Since  $A$  and  $B$  are disjoint, if  $A = B$ , then both must be empty:

$$A = B = \emptyset$$

So there is exactly one fixed pair under swapping, namely  $(\emptyset, \emptyset)$ .

Using symmetry, the number of unordered pairs is

$$\frac{81 + 1}{2} = 41$$

Therefore, the required number is

41

Hence, the correct option is D.

#### COMMON TRAPS

- **Treating unordered pairs as ordered pairs.** For unordered pairs,  $\{A, B\} = \{B, A\}$ ; ordered count must be adjusted for swapping. *Avoid it:* After getting an ordered count, explicitly ask whether the problem says ordered or unordered and whether swapping changes the object.
- **Dividing 81 by 2 without fixed-pair correction.** All non-identical pairs occur in swap pairs, but  $(\emptyset, \emptyset)$  is fixed under swapping and must be counted once. *Avoid it:* Before dividing by 2, find self-swapping cases satisfying  $A = B$  and  $A \cap B = \emptyset$ .

**REVISE** Practise per-element state counting for pairs/triples of sets under conditions like disjoint, union fixed, and equal. • Burnside-style fixed-case corrections for converting ordered counts to unordered counts.

32 Answer: B, C, D

Ref 13389

Hard ~3 min Derivatives and monotonicity

Given function is

$$f(x) = x \cos \frac{1}{x}$$

Differentiate using product rule.

$$f'(x) = \cos \frac{1}{x} + x \cdot \frac{d}{dx} \left( \cos \frac{1}{x} \right)$$

Now,

$$\frac{d}{dx} \left( \cos \frac{1}{x} \right) = -\sin \frac{1}{x} \cdot \left( -\frac{1}{x^2} \right)$$

$$\frac{d}{dx} \left( \cos \frac{1}{x} \right) = \frac{\sin \frac{1}{x}}{x^2}$$

So,

$$f'(x) = \cos \frac{1}{x} + \frac{\sin \frac{1}{x}}{x}$$

For option B, take the limit as  $x \rightarrow \infty$ .

$$\lim_{x \rightarrow \infty} \cos \frac{1}{x} = 1$$

$$\lim_{x \rightarrow \infty} \frac{\sin \frac{1}{x}}{x} = 0$$

Therefore,

$$\lim_{x \rightarrow \infty} f'(x) = 1$$

So option B is correct.

Now check options A and C. Put

$$t = \frac{1}{x}$$

For  $x \geq 1$ , we have

$$0 < t \leq 1$$

Then

$$f'(x) = \cos t + t \sin t$$

Define

$$g(t) = \cos t + t \sin t$$

Differentiate with respect to  $t$ .

$$g'(t) = -\sin t + \sin t + t \cos t$$

$$g'(t) = t \cos t$$

Since  $0 < t \leq 1$ , we have  $\cos t > 0$ , hence

$$g'(t) > 0$$

Also,

$$g(0) = 1$$

Since  $g(t)$  is increasing for  $t > 0$ ,

$$g(t) > 1$$

Therefore,

$$f'(x) > 1$$

for every  $x \geq 1$ .

By the mean value theorem or by integration,

$$f(x+2) - f(x) = \int_x^{x+2} f'(u) du$$

Since  $f'(u) > 1$  for all  $u \geq 1$ ,

$$f(x+2) - f(x) > \int_x^{x+2} 1 du$$

$$f(x+2) - f(x) > 2$$

Thus option C is correct and option A is false.

Now check option D. Differentiate  $f'(x)$ .

$$f'(x) = \cos \frac{1}{x} + \frac{\sin \frac{1}{x}}{x}$$

Differentiating,

$$f''(x) = \frac{\sin \frac{1}{x}}{x^2} + \left( -\frac{\cos \frac{1}{x}}{x^3} - \frac{\sin \frac{1}{x}}{x^2} \right)$$

$$f''(x) = -\frac{\cos \frac{1}{x}}{x^3}$$

For  $x \geq 1$ ,  $0 < \frac{1}{x} \leq 1$ , so

$$\cos \frac{1}{x} > 0$$

Hence,

$$f''(x) < 0$$

Therefore  $f'(x)$  is strictly decreasing on  $[1, \infty)$ .

So option D is correct.

Final answer: B,C,D.

**COMMON TRAPS**

- **Sign error while differentiating  $\cos(1/x)$ .** Correct derivative is  $\sin(1/x)/x^2$ , so  $f'(x) = \cos(1/x) + \sin(1/x)/x$ . *Avoid it:* Before simplifying, explicitly write both factors:  $d(\cos u)/dx = -\sin u \cdot u'$  and  $u' = -1/x^2$ .
- **Confusing the variable substitution  $t=1/x$  and monotonic direction.**  $g(t) = \cos t + t \sin t$  increasing in  $t$  implies  $f'(x) = g(1/x)$  is decreasing in  $x$ , not increasing. *Avoid it:* Write a two-point check:  $x = 1 \Rightarrow t = 1, x \rightarrow \infty \Rightarrow t \rightarrow 0$ .

**REVISE** Chain rule with reciprocal arguments like  $1/x$ . · When substituting  $t = 1/x$ , always map interval endpoints and note order reversal.

**33 Answer: C**

Ref 13239

Medium ~1.5 min

Parity and monotonicity of inverse trigonometric functions

We are given  $g(u) = 2 \tan^{-1}(e^u) - \frac{\pi}{2}$ . First check whether the function is odd or even. Compute  $g(-u)$ . Since  $e^{-u} = \frac{1}{e^u}$  and for  $x > 0$ ,  $\tan^{-1} x + \tan^{-1}(\frac{1}{x}) = \frac{\pi}{2}$ , we get  $\tan^{-1}(e^{-u}) = \frac{\pi}{2} - \tan^{-1}(e^u)$ . Therefore,  $g(-u) = 2 \tan^{-1}(e^{-u}) - \frac{\pi}{2}$ . Substituting the identity,  $g(-u) = 2(\frac{\pi}{2} - \tan^{-1}(e^u)) - \frac{\pi}{2}$ . Hence,  $g(-u) = \frac{\pi}{2} - 2 \tan^{-1}(e^u)$ . Also,  $-g(u) = -(2 \tan^{-1}(e^u) - \frac{\pi}{2})$ . So,  $-g(u) = \frac{\pi}{2} - 2 \tan^{-1}(e^u)$ . Thus,  $g(-u) = -g(u)$ , so  $g$  is odd. Now check monotonicity by differentiating,  $g'(u) = 2 \cdot \frac{e^u}{1+e^{2u}}$ . Since  $e^u > 0$  and  $1 + e^{2u} > 0$  for all real  $u$ , we have  $g'(u) > 0$  for every  $u \in (-\infty, \infty)$ . Therefore,  $g$  is strictly increasing on  $(-\infty, \infty)$ . Hence, the correct option is C.

**COMMON TRAPS**

- **Treating arctan reciprocal identity with wrong sign or missing positivity condition.** Since  $e^u > 0$ , the correct principal-branch identity is  $\tan^{-1}(e^{-u}) = \pi/2 - \tan^{-1}(e^u)$ . *Avoid it:* Before using reciprocal arctan identity, check the sign of the argument. Here  $e^u$  is always positive, so the  $+\pi/2$  version applies.
- **Forgetting to replace every  $u$  by  $-u$  inside the exponential.** Substitution means every occurrence of  $u$  becomes  $-u$ :  $e^u \rightarrow e^{-u}$ , not  $-e^u$  and not unchanged. *Avoid it:* Underline the innermost variable-dependent expression before substitution: here the changing part is  $e^u$ .

**REVISE** Inverse tangent identities with branch conditions, especially reciprocal identities for positive versus negative arguments. · Practice parity checks by explicitly writing  $f(-x)$ , not by judging components like  $e^x$  in isolation.

**34 Answer: B**

Ref 13161

Medium ~1.5 min

Intersection of graphs involving exponential functions

We need to count the number of intersection points of the two curves.

At an intersection point,

$$y = x$$

and

$$y = ke^x$$

So,

$$x = ke^x$$

Since  $e^x > 0$  for every real  $x$ , the sign of  $ke^x$  is the same as the sign of  $k$ .

Given,

$$k \leq 0$$

Case 1:  $k = 0$ .

Then,

$$y = ke^x$$

$$y = 0$$

The line  $y = x$  meets  $y = 0$  only at

$$x = 0$$

So there is exactly one intersection point.

Case 2:  $k < 0$ .

From

$$x = ke^x$$

we can rewrite as

$$xe^{-x} = k$$

Define

$$g(x) = xe^{-x}$$

For  $k < 0$ , the solution must have  $x < 0$ , because  $xe^{-x}$  has the same sign as  $x$ .

Now check monotonicity on  $(-\infty, 0)$ :

$$g'(x) = e^{-x}(1 - x)$$

For  $x < 0$ ,

$$1 - x > 0$$

and

$$e^{-x} > 0$$

Therefore,

$$g'(x) > 0$$

So  $g(x)$  is strictly increasing on  $(-\infty, 0)$ .

Also,

$$g(x) \rightarrow -\infty \text{ as } x \rightarrow -\infty$$

and

$$g(x) \rightarrow 0 \text{ as } x \rightarrow 0^-$$

Thus, for every  $k < 0$ , the equation

$$xe^{-x} = k$$

has exactly one solution.

Combining both cases  $k = 0$  and  $k < 0$ , the curves meet at exactly one point.

Final answer: option B.

#### COMMON TRAPS

- **Dropping the equality case in " $k \leq 0$ ".** " $k \leq 0$ " includes both  $k = 0$  and  $k < 0$ . At  $k = 0$ , the curve " $y = ke^x$ " becomes " $y = 0$ ", and  $y = x$  meets it at exactly  $(0, 0)$ . *Avoid it:* Before solving inequalities with a parameter, split boundary and open interval cases: here write explicitly " $k = 0$ " and " $k < 0$ ".
- **Forgetting that  $e^x$  is always positive.** The solution states "Since  $e^x > 0$  for every real  $x$ , the sign of  $ke^x$  is the same as the sign of  $k$ ." Therefore for  $k < 0$ ,  $ke^x < 0$  for every real  $x$ . *Avoid it:* When an exponential appears, first mark its range:  $e^x \in (0, \infty)$ , never zero or negative.

**REVISE** The range and sign behavior of exponential functions:  $e^x > 0$  for all real  $x$ . Practice parameter boundary cases, especially inequalities involving  $\leq$  or  $\geq$ .

35 Answer: A-p, B-q, C-p, D-s

Ref 13165

Very hard ~5 min Equations involving inverse trigonometric functions

Given equation is

$$\sin^{-1}(ax) + \cos^{-1}(y) + \cos^{-1}(bxy) = \frac{\pi}{2}$$

Use the identity

$$\cos^{-1}(y) = \frac{\pi}{2} - \sin^{-1}(y)$$

So the equation becomes

$$\sin^{-1}(ax) - \sin^{-1}(y) + \cos^{-1}(bxy) = 0$$

Hence

$$\cos^{-1}(bxy) = \sin^{-1}(y) - \sin^{-1}(ax)$$

Taking cosine on both sides,

$$bxy = \cos(\sin^{-1}(y) - \sin^{-1}(ax))$$

Using

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

we get

$$bxy = \sqrt{1 - y^2} \sqrt{1 - a^2 x^2} + axy$$

Therefore

$$\sqrt{(1 - y^2)(1 - a^2 x^2)} = (b - a)xy$$

Now examine each case.

For (A),  $a = 1$  and  $b = 0$ .

$$\sqrt{(1 - y^2)(1 - x^2)} = -xy$$

Squaring,

$$(1 - y^2)(1 - x^2) = x^2 y^2$$

$$1 - y^2 - x^2 + x^2 y^2 = x^2 y^2$$

$$x^2 + y^2 = 1$$

So (A) matches (p).

For (B),  $a = 1$  and  $b = 1$ .

$$\sqrt{(1 - y^2)(1 - x^2)} = 0$$

Thus

$$(1 - y^2)(1 - x^2) = 0$$

So

$$(x^2 - 1)(y^2 - 1) = 0$$

Hence (B) matches (q).

For (C),  $a = 1$  and  $b = 2$ .

$$\sqrt{(1 - y^2)(1 - x^2)} = xy$$

Squaring,

$$(1 - y^2)(1 - x^2) = x^2 y^2$$

$$1 - y^2 - x^2 + x^2 y^2 = x^2 y^2$$

$$x^2 + y^2 = 1$$

So (C) matches (p).

For (D),  $a = 2$  and  $b = 2$ .

$$\sqrt{(1 - y^2)(1 - 4x^2)} = 0$$

Thus

$$(1 - y^2)(1 - 4x^2) = 0$$

This is equivalent to

$$(4x^2 - 1)(y^2 - 1) = 0$$

So (D) matches (s).

Final matching is A-p, B-q, C-p, D-s.

#### COMMON TRAPS

- **Dropping the domain restrictions on all three inverse-trig inputs.** For real inverse trigonometric functions, every argument must be in  $[-1, 1]$ . Also radicals like  $\sqrt{1 - a^2 x^2}$  already encode  $|ax| \leq 1$  and  $\sqrt{1 - y^2}$  encodes  $|y| \leq 1$ . *Avoid it:* Before matching, write  $|ax| \leq 1$ ,  $|y| \leq 1$ ,  $|bxy| \leq 1$  beside each case.
- **Using the wrong identity for cos inverse.** On the principal branches,  $\sin^{-1} t + \cos^{-1} t = \pi/2$  for  $t \in [-1, 1]$ . *Avoid it:* Test the identity with  $y = 0$  :  $\cos^{-1}(0) = \pi/2$ , while  $\pi/2 - \sin^{-1}(0) = \pi/2$ .

**REVISE** Principal ranges of  $\sin^{-1}$  and  $\cos^{-1}$  and their cofunction identity. · Practice deriving  $\cos(\sin^{-1} u - \sin^{-1} v)$  with correct square-root signs.



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